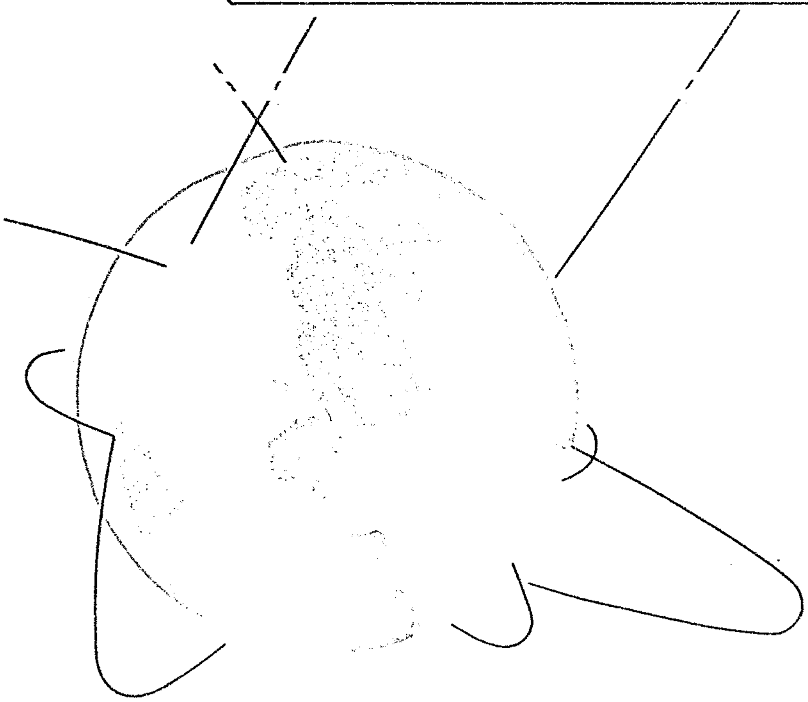
CANONICAL TRANSFORMATION THEORY AND THE OPTIMAL LOW-THRUST PROBLEM

BY

WILLIAM F. POWERS

EMRL 1053

MARCH 1968



ENGINEERING MECHANICS RESEARCH LABORATORY
THE UNIVERSITY OF TEXAS · AUSTIN, TEXAS

CANONICAL TRANSFORMATION THEORY AND THE OPTIMAL LOW-THRUST PROBLEM

by

William F. Powers

Engineering Mechanics Research Laboratory

The University of Texas at Austin

March 1968

This report was prepared under

Grant NGR-44-012-046

for the

National Aeronautics and Space Administration

by the

Engineering Mechanics Research Laboratory

The University of Texas

Austin, Texas

under the direction of

Dr. Byron D. Tapley

Associate Professor and Chairman, Aerospace Engineering

PREFACE

The object of this dissertation is to introduce the methods of canonical transformation theory into the analysis of optimal trajectory problems. Since most optimal trajectory problems can be transformed into Hamiltonian systems which are isomorphic to the Hamiltonian systems of classical mechanics, the theory of canonical transformations is applicable immediately. This study demonstrates applications of both the general theory and the associated perturbation theory.

The Hamilton-Jacobi theory, which is concerned with a particular canonical transformation, is applied to a number of optimal trajectory problems, also. Various techniques for solving the Hamilton-Jacobi equations associated with these problems are presented. Two representative optimal low-thrust missions were used to demonstrate the theory. However, many of the results are applicable to high-thrust analyses as well.

In this dissertation, canonical transformations and the Hamilton-Jacobi theory are viewed as solution techniques. Hopefully, future extensions of the methods presented herein will give good, approximate feedback guidance functions for optimal trajectory problems. Such extensions would give a great impetus to the study of the Caratheodory-Hamilton-Jacobi viewpoint of the calculus of variations, which (in the author's opinion) is the most natural formulation of variational calculus problems.

W. F. P.

January, 1968
Austin, Texas

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to his supervising professor Dr. B. D. Tapley for valuable discussion and critical reading of the manuscript, and for being a constant source of encouragement throughout his graduate study. He would also like to thank his Doctoral Committee, Dr. Lyle G. Clark, Dr. William H. Jefferys, Dr. Walter E. Millett, and Dr. Paul E. Russell, for their time, suggestions, and criticisms.

The author is indebted, also, to the research groups headed by Dr. Rudolf F. Hoelker and Mr. William E. Miner for the education he received while he was member of these groups. The author is especially indebted to Mr. Miner for introducing him to the research area of this dissertation, and for his many helpful discussions throughout this study.

The author wishes to thank Mr. Gil Kim for his able assistance in the generation of the numerical data, and Mrs. Mollie Schluter for her patience and efficient typing of the manuscript.

Finally, special appreciation is reserved for the author's wife and family for their encouragement and understanding during his graduate study.

TABLE OF CONTENTS

Page

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Many of the major problems in modern control theory[34*] and analytical mechanics[1] can be represented by a Hamiltonian system of differential equations. That is, there exists a scalar differentiable function $H(q, p, t)$ such that

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1,\ldots,n) \qquad (1.1)$$

are the ordinary differential equations which describe the given problem. The function H is called a Hamiltonian and the $2n$ variables $q_1,\ldots,q_n$, $p_1,\ldots,p_n$ are called canonic variables. Equations (1.1) are usually referred to as Hamilton's equations or the canonical equations. The theory concerned with transformations between various Hamiltonian representations of the same problem is entitled canonical transformation theory.

Although Lagrange[21] introduced the canonical equations in his theory of perturbations, Hamilton[12] discovered the analogy between Equations (1.1) and certain necessary conditions of the calculus of variations. Since that time, a close association has existed between the calculus of variations and analytical mechanics.

Another of Hamilton's discoveries was the application of a first-order partial differential equation to problems in optics and mechanics. Later, Jacobi[17] extended Hamilton's discovery and showed that under suitable conditions, a complete solution of the first-order partial differential equation

*Numbers indicate references as listed in the Bibliography.

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0 , \tag{1.2}$$

where $p_i \equiv \frac{\partial S}{\partial q_i}$, is equivalent to a general solution of Equations (1.1). Equation (1.2) is called the Hamilton-Jacobi equation, and the equivalence of the solutions of Equations (1.1) and Equation (1.2) is a consequence of Jacobi's Theorem.

Later in the Nineteenth Century, the theory was further advanced by the research of Lie[23] in the transformation theory of differential equations. In particular, he studied the group of contact transformations (which form a subgroup of the group of canonical transformations). In the early Twentieth Century, Caratheodory[6] united the ideas of Cauchy and Pfaff (in partial differential equation theory), Hamilton and Jacobi (in the calculus of variations and partial differential equations), and Lie (in transformation theory). The results of Caratheodory's research form the modern foundations of the Hamilton-Jacobi approach to the calculus of variations.

Since World War II, automation, control, and optimization problems have become increasingly important. Many of these problems require the minimization or maximization of certain variables involved in the physical problem. The classical method for treating such problems is the calculus of variations. However, a straightforward application of the classical calculus of variations requires the admissible control variables to be elements of an open set. Since a linear control problem in which the magnitude of the admissible controls must be less than a specified value requires the control to lie on the boundary of a closed set, modifications and extensions of the classical calculus of variations became necessary. An extension of the classical theory which is applicable to closed control sets was developed by Valentine[45].

Instead of extending the classical theory, some investigators essentially reformulated the variational calculus to meet the requirements of modern control problems. The most notable and rigorous is the maximum principle developed by Pontryagin and his associates[34]. The methods of Pontryagin are similar to the classical calculus of variations, but are more general in that the admissible control region can be a closed set. Also, this method is developed from a Hamiltonian viewpoint, but only Hamilton's equations are involved in the formulation and not the Hamilton-Jacobi equation.

Based on the ideas of Caratheodory, Bellman[2], in 1956, developed an approach (Dynamic Programming) to optimal control problems which involves the Hamilton-Jacobi equation. Since then, Kalman[19,20] has renewed further interest in the Caratheodory viewpoint of the calculus of variations, and Reference 20 presents a discussion of the advantages and disadvantages of this formulation. Other research concerned with the Caratheodory-Hamilton-Jacobi approach has been reported by Hermes[14], De Zur and Haynes[8], and Snow[42]. Also, a text reiterating some of the results of Caratheodory[38], and a complete translation[7] of Caratheodory's major work[6] have been recently published.

The research of References 8, 14, 19, 20, and 42 is mainly concerned with extending the theory of the Caratheodory-Hamilton-Jacobi viewpoint of the calculus of variations to present day problems and discussing the pros and cons of the philosophy of such an approach. This thesis is concerned with the application of general canonical transformation theory, which includes the Hamilton-Jacobi formulation, to a special optimal control problem: the optimal trajectory problem. Since there exists a Hamiltonian function for this problem, it is only natural to expect certain aspects of classical canonical transformation theory to be useful.

In Chapter 2, a consistent outline of canonical transformation theory is presented, and an important necessary and sufficient condition for a canonical transformation is stated and discussed. (The proof of this lengthy theorem isgiven in Appendix A.) Classical discussions of generating functions and Hamilton-Jacobi theory are also presented. The results of a study of the group of homogeneous canonical transformations are given and these indicate important new applications in optimal trajectory analysis. All of the results in this chapter are applicable to high-thrust as well as low-thrust trajectory analysis.

A number of important results are contained in Chapter 3. First a planar optimal trajectory problem is formulated in polar coordinates, and optimality criteria are used to define a new Hamiltonian system which is isomorphic to classical Hamiltonian systems. This formulation is especially convenient for analytic analyses and in the developement of new coordinate systems for numerical integration procedures. As will be shown in both Chapters 3 and 4, the zero-thrust and circumferential thrust cases can be defined by sub-Hamiltonians of the total generalized Hamiltonian.
Since some information about the solutions of the state variable differential equations for these two cases is well known, corresponding knowledge of the associated Lagrange multipliers can be obtained if a certain Pfaff differential equation, defined by the Hamilton-Jacobi equation, can be integrated. The importance of this fact is demonstrated in both Chapters 3 and 4.

The remaining results of Chapter 3 are concerned with the coast-arc problem. Two Hamilton-Jacobi solutions of this problem are presented for the elliptical case in Sections (3.2) and (3.3). Section (3.4) contains the hyperbolic and parabolic coast-arc solutions. In Section (3.5), the feasibility of obtaining approximate feedback guidance functions by a canonical perturbation analysis of the coast-arc solution is discussed.

In Chapter 4, approximate Hamilton-Jacobi solutions are formed for the generalized Hamiltonian defined by the circumferential thrust state variables. The solution of Section (4.1) is a Hamiltonian formulation of classical results, and the state solution can be reduced to the well known Tsien[43] circumferential thrust solution. In Section (4.2), the coast-arc solution of Section (3.3) is perturbed into an approximate circumferential thrust solution which is more general than the solution of Section (4.1). All of the results of this chapter are restricted to trajectories with small instantaneous eccentricities, but this assumption is satisfied on many interesting missions (e.g., circular orbit transfers and the major portion of escape trajectories).

A non-Hamiltonian analysis of the circumferential thrust problem is presented in Chapter 5. This solution is useful for initial estimates in the process of iterating optimal trajectories, and may also serve as a guide in future Hamiltonian analyses of the circumferential thrust problem.

In Chapter 6, important trends which became apparent in the process of numerically converging the optimal trajectories are discussed. The initial Lagrange multipliers and travel times of the converged optimal trajectories are also presented.

The canonic constant relationships developed in Chapters 3 and 4 were evaluated along representative optimal trajectories. The results of this analysis, along with the results of a similar study of the non-Hamiltonian solution developed in Chapter 5, are presented in Chapter 7. The graphical results of these analyses indicate qualitative characteristics of the optimal trajectories. They may also be of use in forming simplifying assumptions in future analytic investigations of the optimal trajectory problem.

Finally, Chapter 8 summarizes the main results of this thesis, and suggestions for future studies are presented.

CHAPTER 2

CANONICAL TRANSFORMATIONS

Most optimal trajectory problems are not integrable in closed-form, so the system of differential equations which define the problem is usually integrated numerically to give the solution. While the numerical solutions are valuable, approximate analytic solutions are needed for a complete understanding of the problem. Since a large class of optimal trajectory problems can be represented by a Hamiltonian system of differential equations, a canonical transformation attack on the analytic problem is suggested. An essential feature of this method is that many times knowledge of half of the integrals of the system leads to full knowledge of the problem without any further integration.

It has only been in the last three years that investigators have published reports on the use of canonical transformations in optimal trajectory analysis. In Reference 36, the use of Hamiltonian perturbation theory in optimal trajectory analysis is discussed and applied to simple problems. The literature on perturbation theory is surveyed also in Reference 36. Since then, Mitchell[26] has applied Hamilton-Jacobi perturbation theory to the high-thrust problem, in much the same way as Nafoosi and Passmore[28]. Also, Fraeijs de Veubeke[10,11] discusses the application of canonical transformation theory to the thrust-coast-thrust problem.

2.1 Basic Transformation Theory

Since there exist several formulations of canonical transformation theory, a consistent outline of the theory will be presented in this section.

Proofs of the various properties can be found in Reference 37; however, two of the more important properties are proved in Appendix A. This development was greatly influenced by Wintner's thorough treatment of the subject in Reference 47.

In this thesis, canonical transformation theory will be developed from the symplectic matrix viewpoint. The canonical matrix, defined below, plays an important role in the symplectic formulation.

Definition 2.1: Let $I_n \equiv n \times n$ identity matrix and $0_n \equiv n \times n$ zero matrix. Then, the $2n \times 2n$ matrix

$$J \equiv \left[\begin{array}{c|c} 0_n & I_n \\ \hline -I_n & 0_n \end{array}\right]$$

is called the canonical matrix.

Properties (of the canonical matrix):

(C.1) $J^2 = -I$, where $I \equiv I_{2n}$.

(C.2) J is nonsingular, i.e., $|J| \neq 0$.

(C.3) $J^{-1} = -J$.

The following definition of a symplectic matrix was used in the investigation presented here.

Definition 2.2: Let M be a $2n \times 2n$ matrix. The matrix M is said to be symplectic if

$$M^T J M = \mu J ,$$

where μ is a nonzero scalar constant.

Siegel[40] and most other texts do not include the constant μ in the definition of a symplectic matrix. However, as Example (2.2) will show, μ is not always equal to +1 in a canonical transformation, as defined in this thesis. Thus, the condition of Definition (2.2) (which will be shown to be necessary and sufficient for a canonical transformation) was termed symplectic in analogy with the classic definition.

Properties (of symplectic matrices):

(S.1) If M is symplectic, then M is nonsingular and $M^{-1} = -\frac{1}{\mu} J M^T J$.

(S.2) Let S be the class of symplectic matrices of order $2n \times 2n$. Then, S is a group with respect to matrix multiplication.

As previously noted, the main reason for studying canonical transformations is because the optimal trajectory problems under consideration can be represented as Hamiltonian systems. Throughout this thesis the following definition will be used for a Hamiltonian system.

Definition 2.3: Let x and λ be n-vectors and t be a scalar parameter. The x_i will be called coordinates (generalized coordinates, states) and the λ_i will be called momenta (generalized momenta, Lagrange multipliers). If there exists a scalar differentiable function $H(x, \lambda, t)$ such that

$$\begin{aligned} \dot{x}_i &\equiv \frac{dx_i}{dt} = \frac{\partial H}{\partial \lambda_i} \\ \dot{\lambda}_i &\equiv \frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i} \end{aligned} \qquad (i = 1,\ldots,n)$$

are the differential equations which describe a given dynamical process, then the set $\{H, x, \lambda\}$ is called a <u>Hamiltonian system</u>.

Notationwise, unless stated otherwise, the variables $\{x, q, Q, \beta, b, B\}$ will represent coordinates, $\{\lambda, p, P, \alpha, a, A\}$ will represent momenta, and $\{H, K\}$ will represent Hamiltonian functions.

<u>Definition 2.4</u>: Let $\{x(q, p, t), \lambda(q, p, t)\} \epsilon C^2$ be a transformation which satisfies the conditions of the implicit function theorem[7] at each point in the domain of interest. If for *every* Hamiltonian $H(x, \lambda, t)$, there exists a scalar function $K(q, p, t)$ such that

$$\dot{q}_i = \frac{\partial K}{\partial p_i} \qquad\qquad (i = 1,\ldots,n)$$
$$\dot{p}_i = -\frac{\partial K}{\partial q_i},$$

then the transformation is said to be <u>canonical</u>.

Note that the word "every" is emphasized in the above definition. The definition does not say that each transformation which preserves Hamiltonian form is canonical, but only those which preserve Hamiltonian form <u>and</u> are independent of the Hamiltonian function. Later, Example (2.1) will demonstrate this fact more clearly.

Definition (2.4) is not a good working definition since one cannot check "every" Hamiltonian function. However, it leads to the following necessary and sufficient condition which is easily implemented. Since the proof of this theorem is not widely available to the engineer, it is given in Appendix A.

Theorem 2.1 (Necessary and Sufficient Condition for a Canonical Transformation): Let $\{x(q, p, t), \lambda(q, p, t)\} \in C^2$ be a transformation which satisfies the conditions of the implicit function theorem, and let M be the Jacobian matrix of the transformation. Then, $\{x(q, p, t), \lambda(q, p, t)\}$ is a canonical transformation if and only if M is a symplectic matrix.

A property of symplectic matrices which is of use in determining whether a transformation is canonical is the following:

Property 2.1: Let $M \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ matrix, where A, B, C, and D are $n \times n$ submatrices. Then, M is a symplectic matrix if and only if A^TC and B^TD are symmetric and $D^TA - B^TC = \mu I$.

This result is proved in Appendix A.

To emphasize the importance of the word every in Definition (2.4), the following example was constructed.

Example 2.1: Consider the transformation from $\{x, \lambda\}$ to $\{q, p\}$ defined by

$$\begin{aligned} x_1 &= p_2 & \lambda_1 &= q_1 \\ x_2 &= p_1 & \lambda_2 &= q_2 . \end{aligned} \tag{2.1}$$

Theorem (2.1) gives an equivalent condition for Definition (2.4). If the Jacobian of the transformation is not symplectic, then the transformation is not canonical. The Jacobian of the transformation defined by Equations (2.1) is

$$M = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

and, hence,

$$M^T J M = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \neq \mu \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] .$$

Thus, M is not symplectic and the transformation is not canonical. However, if the Hamiltonian in the $\{x, \lambda\}$ - system is

$$H = x_1 x_2 ,$$

then

$$K = -\frac{1}{2}(p_1^2 + p_2^2)$$

is a Hamiltonian in the $\{q, p\}$ - system which satisfies the requirements that the Hamiltonian form be preserved. But, if $H = x_1^2$ is the Hamiltonian in the $\{x, \lambda\}$ - system, then there does not exist a function $K(q, p)$ in the $\{q, p\}$ - system which preserves Hamiltonian form. Therefore, the transformation of Equations (2.1) is not independent of the Hamiltonian, as required by Definition (2.4) if the transformation is to be canonical.

The following simple example demonstrates a canonical transformation for which $\mu \neq +1$.

Example 2.2: Consider the one-dimensional transformation which switches the momentum and coordinate without a sign change, i.e., $x = p$ and $\lambda = q$. Then,

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which implies

$$M^T JM = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (-1) \cdot J .$$

The fact that this transformation is canonical can be verified also by defining the new Hamiltonian, K , to be the negative of the old Hamiltonian. That is, let $H(x, \lambda)$ be given. Then, $\dot{x} = \frac{\partial H}{\partial \lambda}$, $\dot{\lambda} = -\frac{\partial H}{\partial x}$. But, $K \equiv -H$ implies

$$\dot{q} = \frac{\partial K}{\partial p} = \frac{\partial(-H)}{\partial x} = \dot{\lambda}$$

$$\dot{p} = \frac{-\partial K}{\partial q} = \frac{\partial(-H)}{\partial \lambda} = \dot{x} ,$$

as desired. Thus, the condition $M^T JM = (+1)J$ is a sufficient condition for a canonical transformation, but not a necessary condition.

Before presenting a condition for a canonical transformation which is of great use in applications, one last point should be made with regard to Theorem (2.1). With reference to Appendix A, if only time independent canonical transformations are considered, then the new Hamiltonian can be determined from the old Hamiltonian by a straightforward substitution of variables. To this end, the following property is presented.

Property 2.2: Let $\{H(x, \lambda, t), x_1,\ldots,x_n, \lambda_1,\ldots,\lambda_n\}$ be a Hamiltonian system. Then,

$$\{H^*(x, \lambda), x_1,\ldots,x_{n+1}, \lambda_1,\ldots,\lambda_{n+1}\} ,$$

with $H^* \equiv H + \lambda_{n+1}$, $x_{n+1} = t$, and $x_{n+1} = 0$ when $t = 0$, is an equivalent Hamiltonian system which does not contain t explicitly, but has $(n+1)$-degrees of freedom with $H^*(x, \lambda)$ as a constant of the motion.

The canonical form of a Hamiltonian system is usually destroyed when the independent variable is changed. However, a simple artifice which employs Property (2.2) can be used to restore a canonical form. Upon application of Property (2.2), $H^*(x, \lambda)$ is a constant of the motion, say

$$H^*(x, \lambda) \equiv h = \text{constant} .$$

Let $\frac{dt}{dy} = \psi(x, \lambda)$, where y is the new independent variable. Then, the new Hamiltonian is

$$\tilde{H} \equiv \psi(x, \lambda) [H^*(x, \lambda) - h]$$

since

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{\partial \psi}{\partial x_i}[H^* - h] + \psi \frac{\partial H^*}{\partial x_i} = \frac{dt}{dy}(-\frac{d\lambda_i}{dt}) = \frac{-d\lambda_i}{dy}$$

$$\frac{\partial \tilde{H}}{\partial \lambda_i} = \frac{\partial \psi}{\partial \lambda_i}[H^* - h] + \psi \frac{\partial H^*}{\partial \lambda_i} = \frac{dt}{dy}\frac{dx_i}{dt} = \frac{dx_i}{dy} .$$

Thus, $\{\tilde{H}, x, \lambda\}$ is a Hamiltonian system with y (instead of t) as the independent variable.

2.2 Generating Functions

Although Theorem (2.1) gives a method for verifying whether or not a transformation is canonical, it does not give a means for determining the transformation itself. Based upon a modified Hamilton's principle, the following theorem presents a sufficient condition for a canonical transformation which is of use in the actual definition of a canonical transformation.

Theorem 2.2: Let $\{H, x, \lambda\}$ and $\{K, q, p\}$ be two Hamiltonian systems. If the Lagrangians

$$L \equiv \sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t)$$

$$L' \equiv \sum_{i=1}^{n} p_i \dot{q}_i - K(q, p, t)$$

in the two systems differ at most by the total time derivative of an arbitrary scalar function, then the transformation $\{x(q, p, t), \lambda(q, p, t)\}$ is a canonical transformation. In other words:

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t) = \sum_{i=1}^{n} p_i \dot{q}_i - K(q, p, t) + \frac{dS^*}{dt} \tag{2.2}$$

is a sufficient condition for the transformation $\{x(q, p, t), \lambda(q, p, t)\}$ to be canonical.

The importance of Equation (2.2) is mainly due to the presence of the scalar function S^*. That is, certain specifications of S^* immediately define canonical transformations, and certain assumptions about the functional form of S^* result in partial differential equations whose solutions define canonical transformations. For these reasons the function S^* is called a <u>generating function</u>.

In many expositions on the subject of canonical transformations only four basic types of generating functions are presented: $S^*(x, q, t)$, $S^*(x, p, t)$, $S^*(\lambda, q, t)$, and $S^*(\lambda, p, t)$. The impression is given that these are the only possible forms. However, there exist many other functional forms for the generating functions, and some of these other forms are extensively used in trajectory analysis. In fact, the four forms mentioned above are used rarely. A rule for developing all forms in which the generating function will depend on n nonconjugate old variables, n nonconjugate new variables, and t (in general) is the following.

<u>Rule 2.1</u>: Let $z_i \in \{x_i, \lambda_i\}$ and $Z_i \in \{q_i, p_i\}$ for each $i = 1, \ldots, n$ (i.e., the nonconjugate condition). Then, the conditions which must be satisfied in order that a canonical transformation with a generating function of the form $S(z, Z, t)$ is defined can be obtained by independence arguments after substituting

$$S^* \equiv S(z, Z, t) + \sum_{i=1}^{n} [x_i \lambda_i (z_i \Delta \lambda_i) - q_i p_i (Z_i \Delta p_i)] \tag{2.3}$$

into Equation (2.2). The Δ-operation is defined by

$$a \Delta b \equiv \begin{cases} 1 & \text{if } a \equiv b \\ 0 & \text{if } a \not\equiv b . \end{cases} \tag{2.4}$$

Throughout this thesis effective use is made of rather simple canonical transformations which involve the switching of coordinates and momenta, a change of sign, and the addition of a constant. A canonical transformation involving, at most, these three operations will be referred to as a simple transformation. A rule for such transformations, to be used in conjunction with Equations (2.2) and (2.3), is the following.

Rule 2.2: Let $p_i \varepsilon \{x_i + a_i, \lambda_i + b_i\}$ for each $i = 1,\ldots,n$, where a_i and b_i are constants. That is, the new momenta are specified to be non-conjugate old variables plus constants. Then, the new coordinates are defined by the conditions obtained by independence arguments after substituting Equation (2.3), with

$$S(z, Z, t) \equiv \sum_{i=1}^{n} [x_i(p_i - b_i)\ (d\lambda_i \,\Delta\, dp_i) - (x_i + a_i)\ q_i(dx_i \,\Delta\, dp_i)], \qquad (2.5)$$

into Equation (2.2). (The reason for specifying the momenta instead of the coordinates in this rule is because the constants of a complete solution of a Hamilton-Jacobi equation are actually momenta variables. This fact will become more evident in the solution of the Hamilton-Jacobi equations of Chapters 3 and 4.)

Since simple transformations are used so freely in this thesis, the following example is given to demonstrate the procedure for applying the above rule.

Example 2.3: Without loss of generality, suppose that

$$\begin{aligned} p_i &= \lambda_i + C_i & \quad (i = 1,\ldots,m < n) \\ p_i &= x_i + C_i\ , & \quad (i = m + 1,\ldots,n) \end{aligned} \qquad (2.6)$$

where the C_i are constants. Application of Equation (2.5) gives

$$S(z, Z, t) \equiv S(x, p_1, \ldots, p_m, q_{m+1}, \ldots, q_n) = \sum_{i=1}^{m} x_i(p_i - C_i) - \sum_{i=m+1}^{n} (x_i + C_i)q_i \; . \tag{2.7}$$

Then, since the generating function has $x_1, \ldots, x_n$, $p_1, \ldots, p_m$, $q_{m+1}, \ldots q_n$ as its independent variables, Equation (2.3) gives

$$S^* = \left[\sum_{i=1}^{m} x_i(p_i - C_i) - \sum_{i=m+1}^{n} (x_i + C_i)q_i \right] - \sum_{i=1}^{m} q_i p_i \; . \tag{2.8}$$

Substitution of Equation (2.8) into Equation (2.2) results in

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i dx_i - Hdt = & \sum_{i=1}^{m} p_i dq_i + \sum_{i=m+1}^{n} p_i dq_i - Kdt \\ & + \sum_{i=1}^{m} [x_i dp_i + (p_i - C_i)dx_i] - \sum_{i=m+1}^{n} [(x_i + C_i)dq_i + q_i dx_i] \\ & - \sum_{i=1}^{m} (q_i dp_i + p_i dq_i) \; . \end{aligned}$$

After cancellation and removal of the $(K - H)dt$ - term:

$$\begin{aligned} & \sum_{i=1}^{m} [(\lambda_i - p_i + C_i)dx_i + (q_i - x_i)dp_i] + \\ & \quad + \sum_{i=m+1}^{n} [(x_i + C_i - p_i)dq_i + (q_i + \lambda_i)dx_i] \equiv 0 \; . \end{aligned}$$

Since the differentials are independent, the simple transformation is defined by:

$$p_i = \lambda_i + C_i \quad ; \qquad q_i = x_i \qquad (i = 1,\ldots,m)$$

$$p_i = x_i + C_i \quad ; \qquad q_i = -\lambda_i \qquad (i = m+1,\ldots,n) \tag{2.9}$$

Equations (2.9) will be used extensively in Chapters 3 and 4.

2.3 Homogeneous Canonical Transformations in Optimal Trajectory Analysis

It is well known in trajectory analysis that if the variational problem under consideration is formulated as a Bolza problem[3] in the calculus of variations, then the generalized Lagrangian is identically zero. For example, in the time-optimal problem the necessary conditions for the quantity

$$I = t_f + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i[\dot{x}_i - f_i(x, u, t)]dt ,$$

(where $\dot{x}_i = f_i(x, u, t)$ are the equations of motion) to be minimized are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1,\ldots,n) \tag{2.10a}$$

$$\frac{\partial}{\partial u}\left[\sum_{i=1}^{n} \lambda_i f_i \right] = 0; \qquad \frac{\partial^2}{\partial u^2}\left[\sum_{i-1}^{n} \lambda_i f_i \right] \leq 0 , \tag{2.10b}$$

for all $t_o \leq t \leq t_f$, and

$$\left[\left(L - \sum_{i=1}^{n} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right)dt + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i} dx_i \right]_{t_o}^{t_f} + dt_f = 0 \tag{2.10c}$$

where

$$L \equiv \sum_{i=1}^{n} \lambda_i[\dot{x}_i - f(x, u, t)] . \tag{2.11}$$

Because of the form of Equations (2.10a), the function, L, defined in Equation (2.11) is called the generalized Lagrangian, which is identically zero by definition. The (generalized) Hamiltonian for this problem is then

$$H \equiv \sum_{i=1}^{n} \lambda_i \dot{x}_i - L$$

or,

$$H = \sum_{i=1}^{n} \lambda_i f_i(x, u, t). \tag{2.12}$$

Later, in this section, it will be shown that the form of Equation (2.12) has important implications in optimal trajectory analysis.

Recall Equation (2.2) of Section (2.2). This equation implies the following differential forms

$$\delta S^* = \sum_{i=1}^{n} (\lambda_i \delta x_i - p_i \delta q_i) \tag{2.13}$$

$$K = \frac{\partial S^*}{\partial t} + H . \tag{2.14}$$

If the transformation under consideration is time independent, then S^* does not depend on time and the two Hamiltonians, K and H, are equal. Further, if the generating function is identically constant, then $\delta S^* \equiv 0$. Such transformations have a special name.

Definition 2.5: A canonical transformation in which $\frac{\partial S^*}{\partial t} = 0$ and $\delta S^* \equiv 0$ is called a homogeneous canonical transformation[7,46] (Mathieu transformation[46]). Furthermore, a homogeneous canonical transformation in which n independent relations between $\{x_1,\ldots,x_n\}$ and $\{q_1,\ldots,q_n\}$ are specified is called an extended point-transformation[46] (extended coordinate transformation[47]).

In Reference 25, a method for transforming Lagrange multipliers from one state-coordinate system to another is presented. This method involves a number of matrix multiplications. Actually, such a transformation is a canonical transformation, and if the Hamiltonian is in the form of Equation (2.12), then the transformation is an extended point-transformation, which greatly simplifies matters. This fact is proved in the following theorem.

Theorem 2.3: Let $x = \phi(q)$ be a nonsingular transformation between the coordinates of two Hamiltonian systems, and let $\dot{x} = f(x, \lambda, t)$ and $\dot{q} = F(q, p, t)$ be the time rates of change of the coordinates. If Hamiltonian systems where $H = \sum_{i=1}^{n} \lambda_i f_i$ and $K = \sum_{i=1}^{n} p_i F_i$ are considered only, then the time independent canonical transformation between the momenta of the two systems is defined by the n-equations

$$p_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial q_i} . \qquad (i = 1,\ldots,n) \tag{2.15}$$

Proof: A sufficient condition for a canonical transformation is given by Equation (2.2), i.e.,

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \frac{dS^*}{dt}$$

or,

$$\sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) = \sum_{i=1}^{n} p_i (\dot{q}_i - F_i) + \frac{dS^*}{dt} .$$

Thus,

$$\frac{dS^*}{dt} \equiv 0 .$$

Since the transformation is time independent

$$\delta S^* \equiv 0 ,$$

which implies, by Equation (2.13):

$$\delta S^* \equiv 0 = \sum_{i=1}^{n} (\lambda_i \delta x_i - p_i \delta q_i) .$$

Application of the given state transformation, $x = \phi(q)$, gives

$$\sum_{i=1}^{n} (\lambda_i \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial q_j} \delta q_j - p_i \delta q_i) = 0$$

or,

$$\sum_{i=1}^{n} (p_i - \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial q_i}) \delta q_i = 0 .$$

Since the q_i $(i = 1,\ldots,n)$ are independent, the coefficients of their variations must be zero, and the desired result is obtained

$$p_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial q_i} . \qquad (i = 1,\ldots,n)$$

Thus, for example, if an optimal trajectory problem is completely formulated in cartesian state coordinates, the Lagrange multiplier transformation between the cartesian system and any other system (e.g., polar coordinates, Poincare variables) is immediately defined by Equations (2.15) and the state variable transformation.

Another type of canonical transformation which is similar to an extended point-transformation is given by the following definition.

Definition 2.6: A time independent canonical transformation in which:

(i) $\delta S \equiv 0$, where

$$S \equiv S^* + \sum_{i=1}^{n} (p_i q_i - x_i \lambda_i) ,$$

and (ii) n independent relations between $\{\lambda_1, \ldots, \lambda_n\}$ and $\{p_1, \ldots, p_n\}$ are specified, is called an extended momenta-transformation.

The reason for defining such a transformation is because of its possible importance in the optimal trajectory problem. Before giving an example of such an application, the corresponding coordinate transformation (which will also contain momenta) will be determined.

Let $\lambda = \psi(p)$ be a nonsingular transformation between the momenta of two Hamiltonian systems. Substitution of the extended momenta-transformation condition (i) into Equation (2.13) gives

$$\sum_{i=1}^{n} (\lambda_i \delta x_i + x_i \delta \lambda_i - p_i \delta q_i - q_i \delta p_i) = \sum_{i=1}^{n} (\lambda_i \delta x_i - p_i \delta q_i)$$

or,

$$\sum_{i=1}^{n} (x_i \delta \lambda_i - q_i \delta p_i) \equiv 0 . \tag{2.16}$$

But, $\lambda = \psi(p)$, so

$$\sum_{i=1}^{n} (x_i \sum_{j=1}^{n} \frac{\partial \psi_i}{\partial p_j} \delta p_j - q_i \delta p_i) \equiv 0$$

or,

$$q_i = \sum_{j=1}^{n} x_j \frac{\partial \psi_j}{\partial p_i} . \quad (i = 1, \ldots, n) \tag{2.17}$$

Example 2.4: Consider an optimal trajectory problem in polar coordinates with thrust and an inverse square gravitational force field. The generalized Hamiltonian is (see Figure 1):

$$H = \lambda_1(v^2/r - k/r^2 + \frac{T}{m}\sin\alpha) + \lambda_2(-uv/r + \frac{T}{m}\cos\alpha) + \lambda_3 u + \lambda_4 v/r \ , \tag{2.18}$$

where $u \equiv \dot{r}$, $v \equiv r\dot{\theta}$, and $m \equiv m_o + \dot{m}_o(t - t_o)$. For a minimum, the following conditions must be satisfied:

$$\frac{\partial H}{\partial \alpha} = \lambda_1 \frac{T}{m}\cos\alpha - \lambda_2 \frac{T}{m}\sin\alpha = 0$$

or,

$$\tan\alpha = \frac{\lambda_1}{\lambda_2} \ , \tag{2.19}$$

and

$$\frac{\partial^2 H}{\partial \alpha^2} = -\lambda_1 \frac{T}{m}\sin\alpha - \lambda_2 \frac{T}{m}\cos\alpha \le 0 \tag{2.20}$$

which implies

$$\cos\alpha = +\lambda_2(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}} \ , \ \sin\alpha = +\lambda_1(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}} \ . \tag{2.21}$$

Note that Equations (2.19) and (2.21) imply that the knowledge of two Lagrange multipliers is necessary to determine one control. On some missions (e.g., multirevolution circular orbit transfers) it is well known that the optimal control behavior is that of a slight oscillation about $\alpha = 0$. Thus, on many of these missions $|\lambda_2| >> |\lambda_1|$. In such cases, it might be more economical to be treating a Hamiltonian system which takes advantage of this knowledge, e.g.,

$$\tan \alpha = p_1 , \tag{2.22}$$

where p_1 is a new multiplier. In this case, the important multiplier is bounded if $|\alpha| < 90°$ (whereas in Equation (2.19) λ_1 and λ_2 may have wide ranges of values with only their ratio bounded), and the control is a function of only one multiplier.

The desired situation, i.e., Equation (2.22), can be obtained by application of an extended momenta-transformation. That is, the following Lagrange multiplier transformation is specified:

$$\begin{aligned} p_1 &= \lambda_1/\lambda_2 & & & \lambda_1 &= p_1 p_2 \\ p_2 &= \lambda_2 & & \leftrightarrow & \lambda_2 &= p_2 \\ p_3 &= \lambda_3 & & & \lambda_3 &= p_3 \\ p_4 &= \lambda_4 & & & \lambda_4 &= p_4 \end{aligned} \tag{2.23}$$

Application of Equations (2.17) requires the new state variables to be

$$\begin{aligned} q_1 &= u\lambda_2 & & & u &= q_1/p_2 \\ q_2 &= v + u\lambda_3/\lambda_4 & & \leftrightarrow & v &= q_2 - q_1 p_1/p_2 \\ q_3 &= r & & & r &= q_3 \\ q_4 &= \theta & & & \theta &= q_4 \end{aligned} \tag{2.24}$$

In the $\{q,p\}$ - system, the optimality criteria of Equations (2.19) and (2.20) requires:

$$\frac{\partial H}{\partial \alpha} = p_1 p_2 \frac{T}{m} \cos \alpha - p_2 \frac{T}{m} \sin \alpha = 0 \tag{2.25}$$

or,

$$\tan \alpha = \frac{p_1 p_2}{p_2} = p_1 ; \tag{2.26}$$

and

$$\frac{\partial^2 H}{\partial \alpha^2} = -p_1 p_2 \frac{T}{m} \sin \alpha - p_2 \frac{T}{m} \cos \alpha \leq 0 . \tag{2.27}$$

From Equation (2.26)

$$\cos \alpha = \pm 1/(1 + p_1^2)^{\frac{1}{2}} , \sin \alpha = \pm p_1/(1 + p_1^2)^{\frac{1}{2}} , \tag{2.28}$$

so Equation (2.27) becomes

$$\frac{\partial^2 H}{\partial \alpha^2} = \frac{T}{m} p_2 \{\pm p_1^2/(1 + p_1^2)^{\frac{1}{2}} \pm 1/(1 + p_1^2)^{\frac{1}{2}}\} \leq 0$$

or,

$$\frac{\partial^2 H}{\partial \alpha^2} = (- \frac{T}{m} p_2/(1 + p_1^2)^{\frac{1}{2}})\{\pm (1 + p_1^2)\} \leq 0 . \tag{2.29}$$

Since $(1 + p_1^2) > 0$ for all real-values of p_1 , the proper sign is determined by the following conditions:

$$\begin{aligned} p_2 &> 0 \quad \leftrightarrow \quad \text{plus sign} \\ p_2 &< 0 \quad \leftrightarrow \quad \text{minus sign} . \end{aligned} \tag{2.30}$$

Since the analysis is only considered feasible for missions on which $|\alpha| < 90°$, then $\cos\alpha > 0$. Thus, from Equations (2.28), the plus sign must be chosen in order to satisfy $\cos\alpha > 0$. This implies that $p_2 > 0$, which is also evident from the fact that

$$\tan\alpha = \frac{p_1 p_2}{p_2} ,$$

i.e., only the numerator can be negative if $|\alpha| < 90°$. Therefore, in summary: the generalized Hamiltonian is

$$\begin{aligned} K = {} & q_1 p_3/p_2 - k p_1 p_2/q_3^2 + (q_2 p_2 - q_1 p_1) \\ & \cdot [p_4 - q_1 + p_1(q_2 p_2 - q_1 p_1)]/q_3 p_2 + \frac{T}{m} p_2 \sqrt{1 + p_1^2} \end{aligned} \tag{2.31}$$

subject to: $\dot{q}_i = \dfrac{\partial K}{\partial p_i}$, $\dot{p}_i = -\dfrac{\partial K}{\partial q_i}$, $(i = 1,\ldots,n)$

and the optimal control is defined by

$$\tan\alpha = p_1 , \quad \cos\alpha = 1/(1 + p_1^2)^{\frac{1}{2}} , \quad \sin\alpha = p_1/(1 + p_1^2)^{\frac{1}{2}} . \tag{2.32}$$

For missions with small control angles (e.g., $|\alpha| \leq 15°$) the functional form of Equation (2.31) is convenient for a binomial expansion of the radical. Note that the approximation $\sqrt{1 + p_1^2} \simeq 1$ represents the circumferential thrust case (i.e., $\alpha \equiv 0$). Also, especially note that each of the relations in Equations (2.32) depend on only one multiplier, p_1 .

With respect to the Bolza problem in the calculus of variations, the above development corresponds to the consideration of a modified generalized Lagrangian defined by

$$L \equiv \sum_{i=1}^{n} g_i(p_1, \ldots, p_n)\,[\dot{x}_i - f(x, u, t)] \;,$$

where the functions $g_1, \ldots, g_n$ represent n-independent functions of n-unknown multipliers. The usual Bolza formulation is a special case defined by $g_i \equiv p_i$. For the Hamiltonian of Example (2.4), the corresponding Lagrangian is

$$L = p_1 p_2(\dot{u} - f_1) + p_2(\dot{v} - f_2) + p_3(\dot{r} - f_3)$$

$$+ \; p_4(\dot{\theta} - f_4) \;.$$

The Euler-Lagrange equation for the control is

$$\frac{\partial L}{\partial \alpha} = -p_1 p_2 \frac{T}{m} \cos \alpha + p_2 \frac{T}{m} \sin \alpha = 0$$

or,

$$\tan \alpha = p_1$$

as expected. Therefore, one can view an extended momenta-transformation as a modified Bolza problem of the calculus of variations in which the multipliers of the equations of motion in the Lagrangian are not required to be n single p_i's .

2.4 Hamilton-Jacobi Theory

The previous sections were concerned with the procedure for performing a canonical transformation when a generating function is given. Attention will now be given to the problem of determining a generating function.

Let $H(x, \lambda, t)$ be a given Hamiltonian system. If a canonical transformation to a new Hamiltonian system where $K \equiv 0$ can be effected, then the integration problem will be trivial, i.e.,

$$\begin{aligned} \dot{q}_i &= \frac{\partial K}{\partial p_i} = 0 & & q_i = \text{constant} \equiv \beta_i \\ & & \longrightarrow & \\ \dot{p}_i &= -\frac{\partial K}{\partial q_i} = 0 & & p_i = \text{constant} \equiv \alpha_i \ . \end{aligned} \tag{2.33}$$

The Hamilton-Jacobi theory has as its fundamental objective the definition of this particular canonical transformation.

Let $\{x, p, t\}$ be the subset of $2n + 1$ independent variables of the set of $4n + 1$ variables $\{x, \lambda, q, p, t\}$. Then, application of Rule (2.1) gives

$$S^* \equiv S(x, p, t) - \sum_{i=1}^{n} q_i p_i \ .$$

Substitution of S^* in Equation (2.2) gives

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H = \sum_{i=1}^{n} p_i \dot{q}_i - K + \frac{dS}{dt} - \sum_{i=1}^{n} (q_i \dot{p}_i + p_i \dot{q}_i)$$

or,

$$\sum_{i=1}^{n} (\lambda_i \dot{x}_i + q_i \dot{p}_i) - H + K - \frac{\partial S}{\partial t} - \sum_{i=1}^{n} \left(\frac{\partial S}{\partial x_i} \dot{x}_i + \frac{\partial S}{\partial p_i} \dot{p}_i\right) \equiv 0 \ .$$

Then, since the set $\{x, p, t\}$ is independent

$$\begin{aligned} \lambda_i &= \frac{\partial S}{\partial x_i} \\ q_i &= \frac{\partial S}{\partial p_i} \qquad\qquad (i = 1,\ldots,n) \\ K &= \frac{\partial S}{\partial t} + H \ . \end{aligned} \tag{2.34}$$

Thus, for the important special case when $K \equiv 0$, the third of Equations (2.34) yields the Hamilton-Jacobi equation (H-J equation):

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0 , \tag{2.35}$$

where the first set of Equations (2.34) has been used to replace λ_i by $\frac{\partial S}{\partial x_i}$ in the Hamiltonian. The H-J equation is a first-order partial differential equation which is to be solved for the generating function $S(x,\alpha,t)$, where $\alpha_i \equiv p_i$ in the $\{K \equiv 0, q_i \equiv \beta_i, p_i \equiv \alpha_i\}$-system. As shown in the following important theorem, if a complete solution of the H-J equation can be determined, then a general solution to the original dynamical problem will be defined.

Theorem 2.3 (Jacobi's Theorem): Let $S(x, \alpha, t)$ be a complete solution of the H-J equation, and $\{\beta\}$ be a set of n arbitrary constants and $\frac{\partial S}{\partial \alpha_i} \equiv \beta_i$. Then, the functions

$$\begin{aligned} x_i &= x_i(\alpha, \beta, t) \\ & \qquad\qquad\qquad\qquad (i = 1,\ldots,n) \\ \lambda_i &= \frac{\partial S(x, \alpha, t)}{\partial x_i} \equiv \lambda_i[x(\alpha, \beta, t), \alpha, t] \end{aligned}$$

constitute a general solution of the original Hamilton's equations, i.e.,

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} , \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} . \qquad (i = 1,\ldots,n)$$

In astronomy and atomic physics, most of the Hamiltonian systems encountered are conservative (i.e., time does not appear explicitly in the Hamiltonian function). In this case, $H(x, \lambda)$ is a constant of the motion since

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \dot{x}_i + \frac{\partial H}{\partial \lambda_i} \dot{\lambda}_i \right) = \sum_{i=1}^{n} (-\dot{\lambda}_i \dot{x}_i + \dot{x}_i \dot{\lambda}_i) \equiv 0 .$$

Thus, for such problems it is sometimes more convenient to consider generating functions which do not depend on time. Then, from Equations (2.34)

$$\begin{aligned} \lambda_i &= \frac{\partial S(x, p)}{\partial x_i} \\ & \qquad\qquad (i = 1,\ldots,n) \\ q_i &= \frac{\partial S(x, p)}{\partial p_i} \qquad\qquad (2.36) \\ K(q, p) &= H[x(q, p), \lambda(q, p)] . \end{aligned}$$

Instead of requiring $K \equiv 0$, let the new Hamiltonian be any specified function of the new momenta, i.e., $K = K(p)$. Since K is a Hamiltonian

$$\begin{aligned} \dot{p}_i &= -\frac{\partial K}{\partial q_i} = 0 \\ & \qquad\qquad (i = 1,\ldots,n) \\ \dot{q}_i &= \frac{\partial K}{\partial p_i} . \end{aligned}$$

Thus, $p_i = \text{constant} \equiv \alpha_i$ $(i = 1,\ldots,n)$ and so

$$\dot{q}_i = \frac{\partial K(\alpha)}{\partial \alpha_i} = \text{constant} .$$

Hence, once again the integration problem is trivial and the last of Equations (2.36) becomes

$$H(x, \frac{\partial S}{\partial x}) = K(\alpha) . \qquad (2.37)$$

A special case of Equation (2.37) is

$$H(x, \frac{\partial S}{\partial x}) = \alpha_1 ,$$

which Born[5] refers to as the Hamilton-Jacobi equation.

As it stands, the Hamilton-Jacobi theory is elegant but it cannot be used to solve many problems since it involves the integration of a partial differential equation. However, approximate solutions to many nonlinear problems have been obtained by the application of perturbation theories based on the H-J equation.

There are two basic techniques for obtaining perturbation solutions with Hamilton-Jacobi theory: attack a new set of ordinary differential equations (canonical perturbation theory) or attack a new partial differential equation (H-J perturbation theory). Instead of developing the perturbation procedure for these two methods separately, they will be derived together since the derivations are essentially the same. Moreover, when these two techniques are applied it may be advantageous to use a combination of the two. The basic idea in both procedures is to make the integration problem trivial by performing a sequence of transformations which converge to "natural" variables for the problem (e.g., a set of canonic constants).

Let $\{H(x, \lambda, t), x, \lambda\}$ be a Hamiltonian system. Suppose that the Hamiltonian is partitioned as

$$H = H_o - \sum_{i=1}^{m} H_i ,$$

where a complete solution of the H-J equation for H_o is known. In practice, the finite sum is sometimes replaced by an infinite sum (e.g., a power series or Fourier series expansion for $H - H_o$), but the procedure is the same as for a finite sum.

Since the H-J theory assumes the set $\{x, p, t\}$ is the independent subset of $\{x, \lambda, q, p, t\}$, then the general equations for a canonical transformation are Equations (2.34), i.e.,

$$\begin{aligned} \lambda_i &= \frac{\partial S}{\partial x_i} \\ q_i &= \frac{\partial S}{\partial p_i} \qquad (i = 1,\ldots,n) \\ K &= \frac{\partial S}{\partial t} + H = \left(\frac{\partial S}{\partial t} + H_o \right) - \sum_{i=1}^{m} H_i \ . \end{aligned} \tag{2.38}$$

Let $S^o(x, \alpha, t)$ be a complete solution of the H-J equation for H_o,

$$\frac{\partial S^o}{\partial t} + H_o(x, \frac{\partial S^o}{\partial x}, t) = 0 \ , \tag{2.39}$$

and let the system

$$\begin{aligned} x_i^o &= x_i^o(\alpha, \beta, t) \\ & \qquad\qquad (i = 1,\ldots,n) \\ \lambda_i^o &= \lambda_i^o(\alpha, \beta, t) \end{aligned}$$

be the general solution of Hamilton's equations for H_o, where the set $\{\alpha, \beta\}$ is the set of canonic constants determined by the solution of Equation (2.39). From the last of Equations (2.38)

$$K = 0 - \sum_{i=1}^{m} H_i \ ,$$

or

$$K(\alpha, \beta, t) \equiv -\sum_{i=1}^{m} H_i[x^o(\alpha, \beta, t), \lambda^o(\alpha, \beta, t), t] \tag{2.40}$$

and

$$\dot{\beta}_i = \frac{\partial K}{\partial \alpha_i} \qquad \dot{\alpha}_i = -\frac{\partial K}{\partial \beta_i} . \qquad (i = 1,\ldots,n) \tag{2.41}$$

Thus, the result is a new Hamiltonian system $\{K, \alpha, \beta\}$.

There are two basic ways of attacking the "new" Hamiltonian problem defined by Equations (2.40) and (2.41). Canonical perturbation theory involves the integration of Equations (2.41) whereas H-J perturbation theory involves the integration of the H-J equation for the Hamiltonian K .

Define $K_o(\alpha, \beta, t) \equiv -H_1[x^o(\alpha, \beta, t), \lambda^o(\alpha, \beta, t), t]$. Then,

$$K = K_o - \sum_{i=2}^{m} H_i .$$

Consider the H-J equation for K_o :

$$\frac{\partial S^1}{\partial t} + K_o(\beta, \frac{\partial S^1}{\partial \beta}, t) = 0 . \tag{2.42}$$

Let $S^1(a, \beta, t)$ be a complete solution of Equation (2.42). Applying the general canonical transformation Equations (2.38) again leads to the following expressions

$$\alpha_i = \frac{\partial S^1}{\partial \beta_i}$$

$$b_i = \frac{\partial S^1}{\partial a_i} \qquad (i = 1,\ldots,n)$$

$$K^* = \frac{\partial S^1}{\partial t} + K = \left(\frac{\partial S^1}{\partial t} + K_o\right) - \sum_{i=2}^{m} H_i = -\sum_{i=2}^{m} H_i .$$

The set $\{a, b\}$ is a set of canonic constants for the problem defined by $H_o - H_1$, and from Jacobi's theorem, the set of equations

$$\begin{aligned} \beta_i &= \beta_i(a, b, t) \\ \alpha_i &= \alpha_i(a, b, t) \end{aligned} \qquad (i = 1,\ldots,n)$$

represents a general solution of the Hamilton's equations, i.e., Eqs. (2.41). If $H_o - H_1$ is a valid approximation to the total Hamiltonian H, then the system

$$\begin{aligned} x_i^1(a, b, t) &\equiv x^o[\alpha(a, b, t), \beta(a, b, t), t] \\ \lambda_i^1(a, b, t) &\equiv \lambda^o[\alpha(a, b, t), \beta(a, b, t), t] \end{aligned} \qquad (i = 1,\ldots,n)$$

should be a valid approximation to the general solution of the Hamilton's equations for the total Hamiltonian.

If the effects of $\sum_{i=2}^{m} H_i$ are required, the same procedure can be applied to H_2, H_3, etc. One of the most powerful aspects of a Hamiltonian perturbation theory is that one need not start all over when a higher order approximation or the effect of a new perturbation is required.

CHAPTER 3

THE COAST-ARC PROBLEM

In many optimal trajectory problems it is sometimes desirable to allow a coasting period (i.e., zero-thrust) during the trajectory[4,35]. For example, in a circular oribt rendezvous problem, if the final mass of the vehicle is to be maximized, a gain in the final mass is sometimes accomplished by allowing a coast-period in the trajectory. Such missions motivated interest in the coast-arc problem.

Definition 3.1: Let $\dot{x} = f^*[x, \frac{T}{m}\psi(x,\lambda,t),t]$ be the equations of motion and $\dot{\lambda} = g(x, \lambda, t)$ be the Euler-Lagrange equations (of the multipliers) for an optimal trajectory problem where $u = \psi(x, \lambda, t)$ has been obtained by some optimality criteria. The problem defined by the differential equations:

$$\dot{x} = f^*[x, 0, t] \equiv f(x, t)$$

$$\dot{\lambda} = g(x, \lambda, t)$$

is called the coast-arc problem.

The planar coast-arc problem was first solved by Miner[25] and then by Eckenwiler[9], Hempel[13], and Ng and Palmadesso[29]. The solutions of References 9, 13, and 25 treat circular conditions as a special case and not as a continuous extension of the elliptical case. These solutions also involve the integration of the full set of ordinary differential equations. In Section (3.2), a solution of the coast-arc problem is obtained by application of the Hamilton-Jacobi method. This solution is essentially the same as those mentioned above except that it has the advantage of being defined by canonic constants instead of ordinary constants of integration.

The solution of Reference 29 does not require a special solution for circular conditions, but it does involve numerous integrations. In Section (3.3), a solution of the coast-arc problem is obtained by first performing a canonical transformation of the original canonical system of Section (3.2). This results in a separable Hamilton-Jacobi equation whose solution involves the evaluation of only one indefinite integral. This solution does not require a special case for circular conditions, and it also has the advantage of being defined by canonic constants instead of ordinary constants of integration, such as those involved in the solution of Reference 29.

The interest in the coast-arc problem in this thesis is not for its application to coasting portions of trajectories, but rather for its potential as a base solution in a canonical perturbation analysis of the low-thrust problem. For example, the generalized Hamiltonian for many optimal trajectory problems is of the form

$$H = H_o + \frac{T}{m} H_1 ,$$

where H_o defines the coast-arc problem. Thus, for missions where the gravitational forces are dominate when compared with the thrust-forces (e.g., near-planet, low-thrust missions), the possibility exists of perturbing the coast-arc solution into an approximate general solution to the total problem.

It should be noted that the solution of Section (3.3) appears to be preferable in every way to the solution of Section (3.2) since it does not have either a circular singularity or the ± sign difficulties, and it can be solved by separation of variables. However, the solution of Section (3.2) was obtained before the solution of Section (3.3), and its study eventually lead to the solution of Section (3.3). Further, the method of solution presented in Section (3.2) demonstrates the feasibility of attacking nonseparable Hamilton-Jacobi equations.

3.1 Development of a Classical Hamiltonian Form

Before the classical Hamiltonian theory can be applied in trajectory analysis, the optimal trajectory problem must be expressed as a well-defined Hamiltonian system of first-order ordinary differential equations defined by a Hamiltonian function and a set of $2n + 2$ boundary conditions.

Consider the problem of extremizing the function

$$I = G(x_f, t_f) \tag{3.1}$$

subject to the constraints

$$\dot{x}_i - f_i(x, u, t) = 0 , \qquad (i = 1,\ldots,n) \tag{3.2}$$

and the geometric boundary conditions

$$x_i(t_o) = X_{io} \qquad (i = 1,\ldots,n) \tag{3.3}$$

$$M_i(x_f, t_f) = 0 , \qquad (i = 1,\ldots,p\leq n) \tag{3.4}$$

where x is a n-vector of state variables and u is a m-vector of control variables. The problem can be formulated as a Bolza problem[3] in the calculus of variations by introducing a set of unknown multipliers $\{\lambda_1,\ldots,\lambda_n\}$ and forming the augmented functional

$$I = G(x_f, t_f) + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i[\dot{x}_i - f_i(x, u, t)]dt . \tag{3.5}$$

If I is to be an extremal with respect to the choice of $u(t)$, the following necessary conditions must be satisfied:

(i) Lagrange's equations must be satisfied everywhere in the interval $t_o \leq t \leq t_f$, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1,\ldots,n) \tag{3.6}$$

$$\frac{\partial L}{\partial u_i} = 0 , \qquad (i = 1,\ldots,m) \tag{3.7}$$

where

$$L \equiv \sum_{i=1}^{n} \lambda_i[\dot{x}_i - f_i(x, u, t)] ; \tag{3.8}$$

and

(ii) a set of transversality conditions, say:

$$N_i(x_f, \lambda_f, t_f) = 0 , \qquad (i = 1,\ldots,n-p+1) \tag{3.9}$$

must be satisfied at the terminal time.

The system of Equations (3.2), (3.6), and (3.7) can be expressed as a system of first-order equations by defining a generalized Hamiltonian function

$$H^*(x, \lambda, u, t) \equiv \sum_{i=1}^{n} \lambda_i\dot{x}_i - L(x, \dot{x}, \lambda, u, t) , \tag{3.10}$$

and then developing Hamilton's equations

$$\dot{x}_i = \frac{\partial H^*}{\partial \lambda_i} \qquad \dot{\lambda}_i = -\frac{\partial H^*}{\partial x_i} . \quad (i = 1,\ldots,n) \tag{3.11}$$

Equations (3.7) and (3.10) can be combined to yield

$$\frac{\partial H^*}{\partial u_i} = 0 . \qquad (i = 1,\ldots,m) \tag{3.12}$$

In addition, the Weierstrass condition[3] must be satisfied if the functional defined by Equation (3.5) is to be a minimum. This leads to the further requirement that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 H^*}{\partial u_i \partial u_j} \delta u_i \delta u_j \leq 0 , \tag{3.13}$$

for all admissible δu_i and δu_j .

In most optimal trajectory problems, Equations (3.12) and (3.13) can be used to express the control variables as functions of the state variables and the Lagrange multipliers, say

$$u_i = g_i(x, \lambda) . \qquad (i = 1,\ldots,m) \tag{3.14}$$

Consider the composite function

$$H(x, \lambda, t) \equiv H^*[x, \lambda, g(x, \lambda), t] , \tag{3.15}$$

and its partial derivatives with respect to the x_i and λ_i

$$\begin{aligned} \frac{\partial H}{\partial x_i} &= \frac{\partial H^*}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial H^*}{\partial u_j} \frac{\partial u_j}{\partial x_i} \\ & & (i = 1,\ldots,n) \\ \frac{\partial H}{\partial \lambda_i} &= \frac{\partial H^*}{\partial \lambda_i} + \sum_{j=1}^{m} \frac{\partial H^*}{\partial u_j} \frac{\partial u_j}{\partial \lambda_i} . \end{aligned} \tag{3.16}$$

By Equation (3.12), $\frac{\partial H^*}{\partial u_j} = 0$ so Equations (3.16) reduce to

$$\begin{aligned} \frac{\partial H}{\partial x_i} &= \frac{\partial H^*}{\partial x_i} = -\dot{\lambda}_i \\ & & (i = 1,\ldots,n) \\ \frac{\partial H}{\partial \lambda_i} &= \frac{\partial H^*}{\partial \lambda_i} = \dot{x}_i . \end{aligned} \tag{3.17}$$

Thus, Equations (3.17) are Hamilton's equations for the Hamiltonian of Equation (3.15), which is in the classical form since it does not contain any variables which do not have conjugate momenta (e.g., the control variables).

Equations (3.17) and the boundary conditions of Equations (3.3), (3.4), and (3.9) represent a well-defined Hamiltonian system described by the "generalized coordinates" $\{x_1, \ldots, x_n\}$ and the "generalized momenta" $\{\lambda_1, \ldots, \lambda_n\}$. Thus, the classical perturbation theories of Hamiltonian mechanics are now available for the optimal trajectory problem.

3.2 A Natural Polar Base Solution

Consider the problem of minimizing the time of flight of a vehicle powered by a continuously-thrusting engine in which the thrust and mass flow-rate are assumed constant. The only other forcing effect is due to an inverse square gravitational field of a nonrotating spherical body. The motion is assumed to take place in a single plane, and the state of the vehicle will be described by a polar coordinate system. The control variable is the angle between the thrust vector and a line perpendicular to the radius vector (see Figure 1).

Let $u \equiv \dot{r}$ and $v \equiv r\dot{\theta}$. Then, the equations of motion for the vehicle are

$$\begin{aligned} \dot{u} &= v^2/r - k/r^2 + \frac{T}{m}\sin\alpha & \dot{r} &= u \\ \dot{v} &= -uv/r + \frac{T}{m}\cos\alpha & \dot{\theta} &= v/r \end{aligned} \tag{3.18}$$

$$m \equiv m_o + \dot{m}_o(t - t_o) \; . \tag{3.19}$$

By introducing four unknown Lagrange multipliers and referring to the right-hand sides of Equations (3.18) as $f_i(x, \alpha, t)$ (where $x_1 = u$, $x_2 = v$, $x_3 = r$, $x_4 = \theta$), a generalized Hamiltonian can be defined

$$H^* \equiv \sum_{i=1}^{4} \lambda_i f_i(x, \alpha, t) . \tag{3.20}$$

The control, α , can be expressed as a function of Lagrange multipliers by applying Equation (3.12) and (3.13) to the Hamiltonian of Equation (3.20), i.e.,

$$\cos\alpha = +\lambda_2(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}} \qquad \sin\alpha = +\lambda_1(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}} . \tag{3.21}$$

Upon substitution of Equations (3.21) into Equation (3.20), a new Hamiltonian is determined

$$H(x, \lambda, t) = H^*[x, \lambda, \alpha(\lambda), t]$$

or, in explicit form

$$H = [\lambda_1(v^2/r - k/r^2) - \lambda_2 uv/r + \lambda_3 u + \lambda_4 v/r] + \frac{T}{m}(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}} . \tag{3.22}$$

Equation (3.22) can be partitioned into a base Hamiltonian H_o and a perturbing term $\frac{T}{m}(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}$, e.g.,

$$H \equiv H_o + \frac{T}{m}(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}} . \tag{3.23}$$

If the thrust, T , is zero, then $H = H_o$ defines the coast-arc problem. A complete solution of the H-J equation for H_o will now be obtained.

Before writing the H-J equation for H_o , it will be advantageous to consider a physical interpretation of the problem. Since thrust is zero, the state variables represent a Keplerian orbit and thus, are not affected by the Lagrange multipliers. (This fact is also evident by inspection of the equations of Definition (3.1).) Hence, for the base problem, the energy and angular momentum of the orbit should be constants of the motion. Also, H_o does not contain time explicitly, so H_o should be a constant of the motion; and θ does not appear explicitly in H_o, so the conjugate variable of θ should be a constant of the motion. Thus, four constants of the motion are readily apparent. To apply the H-J theory most effectively, a simple canonical transformation should be performed in such a way that the above mentioned constants of the motion can be related to four momenta variables of the base Hamiltonian. For example, the energy and angular momentum integrals are strictly functions of state variables (i.e., generalized coordinates), so a simple transformation should be employed to make two of the state variables, in these relations, generalized momenta. Then, the four constants of the complete solution to the H-J equation can be defined and the only remaining problem will be the integration of a Pfaff differential equation which defines the generating function.

The arguments given above can be implemented by performing the simple canonical transformation resulting from the following specification of new momenta

$$p_1 = u , \quad p_2 = v , \quad p_3 = \lambda_3 , \quad p_4 = \lambda_4 . \tag{3.24}$$

From Example (2.3), the new coordinates are

$$q_1 = -\lambda_1 , \quad q_2 = -\lambda_2 , \quad q_3 = r , \quad q_4 = \theta . \tag{3.25}$$

Since time is not involved in the transformation, the new Hamiltonian, H_o', is found by direct substitution

$$H_o' = -q_1(p_2^2/q_3 - k/q_3^2) + q_2p_1p_2/q_3 + p_1p_3 + p_2p_4/q_3 \; . \tag{3.26}$$

The H-J equation for H_o', with $p_i \equiv \frac{\partial S}{\partial q_i}$, is

$$\frac{\partial S}{\partial t} + H_o'(q, \frac{\partial S}{\partial q}) = 0 \; . \tag{3.27}$$

Since t and q_4 each appear only once in Equation (3.27) and only in the form of a partial derivative with respect to S, it is reasonable to assume a partial separation of variables, i.e.,

$$S = S_1(t) + S_2(q_4) + S^*(q_1, q_2, q_3) \; . \tag{3.28}$$

By substituting Equation (3.28) into Equation (3.27) and using independence arguments, the following expressions are obtained

$$\frac{\partial S}{\partial t} = \frac{\partial S_1}{\partial t} = \alpha_1 \; , \quad \frac{\partial S}{\partial q_4} = \frac{\partial S_2}{\partial q_4} = \alpha_2 \; , \tag{3.29}$$

where α_1 and α_2 are constants. Thus,

$$S = \alpha_1 t + \alpha_2 q_4 + S^*(q_1, q_2, q_3) \; , \tag{3.30}$$

and the H-J equation is reduced to an equation involving only three independent variables.

For a complete solution of Equation (3.27), two more constants must be obtained. As previously mentioned, knowledge of the two-body problem can be used to define the other two, i.e., the energy and angular momentum of the orbit. However, for some problems the knowledge of integrals may not be known beforehand. Thus, the following procedure may be used to attempt to find the remaining unknown constants. (In Chapter 4 such a situation occurs.)

Consider the characteristic system[41] for the H-J equation, and in particular, $\frac{dq_3}{d\tau}$ and $\frac{dp_2}{d\tau}$ (where τ is an arbitrary parameter)

$$\frac{dq_3}{d\tau} = p_1 , \qquad \frac{dp_2}{d\tau} = -p_1 p_2/q_3 .$$

These equations can be combined to give

$$\frac{dp_2}{dq_3} = -\frac{p_2}{q_3}$$

which integrates to give the angular momentum integral

$$\alpha_3 = q_3 p_2 . \tag{3.31}$$

To obtain the energy integral, Equation (3.31) is used along with the characteristic equations for $\frac{dq_3}{d\tau}$ and $\frac{dp_1}{d\tau}$, i.e.,

$$\frac{dq_3}{d\tau} = p_1 , \qquad \frac{dp_1}{d\tau} = \alpha_3^2/q_3^3 - k/q_3^2 .$$

These equations can be combined to give

$$p_1 \frac{dp_1}{dq_3} = \alpha_3^2/q_3^3 - k/q_3^2$$

which integrates to give the energy integral

$$\alpha_4 = -p_1^2 - \alpha_3^2/q_3^2 + 2k/q_3 ,$$

where α_4 is taken to be the negative of energy for convenience.

Thus, the four constants required for the complete solution are defined. The only remaining problem is to incorporate them into the generating function of Equation (3.30). This can be effected by considering the following integrable Pfaff differential equation

$$dS^* = \frac{\partial S^*}{\partial q_1} dq_1 + \frac{\partial S^*}{\partial q_2} dq_2 + \frac{\partial S^*}{\partial q_3} dq_3 , \tag{3.33}$$

where

$$\frac{\partial S^*}{\partial q_1} = p_1(q_3) = \pm(-\alpha_3^2 + 2kq_3 - \alpha_4 q_3^2)^{\frac{1}{2}}/q_3 \tag{3.34}$$

$$\frac{\partial S^*}{\partial q_2} = p_2(q_3) = \alpha_3/q_3 \tag{3.35}$$

and where $\frac{\partial S^*}{\partial q_3}(q_1, q_2, q_3)$ is defined by substituting Equations (3.34) and (3.35) into Equation (3.27); that is, $\frac{\partial S^*}{\partial q_3}$ is defined by satisfaction of the H-J equation. Since Equations (3.34) and (3.35) only depend upon q_3, then S^* is necessarily of the form

$$S^* = p_1(q_3)q_1 + p_2(q_3)q_2 + S'(q_3) , \tag{3.36}$$

for otherwise $\frac{\partial S^*}{\partial q_1}$ and $\frac{\partial S^*}{\partial q_2}$ would depend on variables other than q_3, which is contradictory to the functional form of Equations (3.34) and (3.35). (See pp. 67-68 of Reference 36 for a more thorough discussion of this property.)

Note that Equation (3.34) involves a $\pm$ sign. This corresponds to the radius increasing from perigee to apogee ($p_1 \equiv \dot{r} \geq 0$) and decreasing from apogee to perigee ($p_1 \equiv \dot{r} \leq 0$).

To determine $S'(q_3)$ of Equation (3.36), consider $\frac{\partial S^*}{\partial q_3}$ as defined by the H-J equation. Since neither q_1 nor q_2 can appear in S', then

$$S' = \int \left(\frac{\partial S^*}{\partial q_3}\right)' dq_3 , \tag{3.37}$$

where $\left(\frac{\partial S^*}{\partial q_3}\right)'$ is that portion of $\frac{\partial S^*}{\partial q_3}$ which does not contain q_3 or q_4. Integration of Equation (3.37) yields

$$\begin{aligned} S'(q_3) = {} & \alpha_1 q_3 p_1/\alpha_4 \pm k\alpha_1\alpha_4^{-3/2}\sin^{-1}[(k - \alpha_4 q_3) \cdot \\ & \cdot (k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}] \pm \alpha_2 \sin^{-1}[(kq_3 - \alpha_3^2)(k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}/q_3] . \end{aligned} \tag{3.38}$$

The evaluation of the indefinite integral of Equation (3.37) involves an assumption about the sign of α_4 (the negative of the energy) and $\alpha_4 > 0$ (the elliptic case) was assumed for the result of Equation (3.38). By assuming $\alpha_4 < 0$ and then integrating Equation (3.37), the hyperbolic case is defined. To form the parabolic case ($\alpha_4 = 0$), the $\beta_4 = \frac{\partial S}{\partial \alpha_4}$ expression should be formed before setting $\alpha_4 = 0$. Otherwise, $\beta_4 \equiv 0$ would be obtained and this gives no information. In Section 3.4, the hyperbolic and parabolic solutions are determined for the formulation of Section (3.3).

Consideration of Equations (3.30), (3.36), and (3.38) gives the generating function for the base solution

$$\begin{aligned} S = {} & \alpha_1 t + \alpha_2 q_4 + p_1(q_3, \alpha_3, \alpha_4)q_1 + \\ & + p_2(q_3, \alpha_3, \alpha_4)q_2 + S'(q_3, \alpha_1, \alpha_2, \alpha_3, \alpha_4) . \end{aligned} \tag{3.39}$$

By Jacobi's Theorem, the remaining canonic constants of the motion are obtained by differentiating the generating function with respect to each of the α_i's , i.e., $\beta_i \equiv \frac{\partial S}{\partial \alpha_i}$. After eliminating the intermediate variables, $\{p, q\}$, by Equations (3.24) and (3.25), the complete set of canonic constants for the coast-arc are:

$$
\begin{aligned}
\alpha_1 &= -\lambda_1(v^2/r - k/r^2) + \lambda_2 uv/r - \lambda_3 u - \lambda_4 v/r \\
\alpha_2 &= \lambda_4 \\
\alpha_3 &= rv \\
\alpha_4 &= -u^2 - v^2 + 2k/r \\
\beta_1 &= t + ru/\alpha_4 + \frac{u}{|u|} k\alpha_4^{-3/2} \sin^{-1}[(k - \alpha_4 r) \cdot \\
&\quad \cdot (k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}] \\
\beta_2 &= \theta - \frac{u}{|u|} \sin^{-1}[(kr - \alpha_3^2)(k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}/r] \qquad (3.40) \\
\beta_3 &= -\lambda_2/r + \alpha_3\lambda_1/(r^2u) + \lambda_4/(ru) + \\
&\quad + [\alpha_1\alpha_3(\alpha_3^2 - kr) + k\lambda_2(k - \alpha_4 r)]\,[ru(k^2 - \alpha_4\alpha_3^2)]^{-1} \\
\beta_4 &= \lambda_1/(2u) - \alpha_1 r(\alpha_4 + 2u^2)/(2u\alpha_4^2) - \\
&\quad - \frac{3}{2}\frac{u}{|u|}\alpha_1 k\alpha_4^{-5/2} \sin^{-1}[(k - \alpha_4 r)(k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}] \\
&\quad + [k\alpha_1(\alpha_3^2\alpha_4 r - 2k^2 r + k\alpha_3^2) + \alpha_2\alpha_3\alpha_4^2(\alpha_3^2 - kr)] \cdot \\
&\quad \cdot [2\alpha_4^2 ru(k^2 - \alpha_4\alpha_3^2)]^{-1}.
\end{aligned}
$$

The set $\{\alpha, \beta\}$ represents the closed-form solution to the coast-arc problem. The subset $\{\alpha_3, \alpha_4, \beta_1, \beta_2\}$ defines the Keplerian orbit, and the subset $\{\alpha_1, \alpha_2, \beta_3, \beta_4\}$ defines the solution for the Lagrange multipliers on a coast-arc. However, Equations (3.40) possess two singularities: $\dot{r} \equiv u = 0$ and zero-energy (i.e., $\alpha_4 = 0$). Thus, the class of missions to which the set $\{\alpha, \beta\}$ is applicable is somewhat restricted (i.e., instantaneous elliptical conditions with $\dot{r}$ either positive or negative for the entire trajectory). In Section 3.3, the $\dot{r} = 0$ and $\pm$ sign restrictions are overcome.

The algebraic inversion of Equations (3.40), which is necessary to define $\{x(\alpha, \beta, t), \lambda(\alpha, \beta, t)\}$, is not possible since the β_1-equation is a form of Kepler's equation. However, the canonical perturbation equations due to thrust, for the α_i's and β_i's can still be determined by application of the implicit function theorem [7]. That is, the β_1-equation defines an implicit relationship

$$r = \phi(t, \alpha_3, \alpha_4, \beta_1) ,$$

so the perturbing Hamiltonian is

$$H_1 = K[\alpha, \beta, \phi(t, \alpha_3, \alpha_4, \beta_1)] \equiv \frac{T}{m}[\lambda_1^2(\alpha, \beta, r) + \lambda_2^2(\alpha, \beta, r)]^{\frac{1}{2}} .$$

The canonic perturbation equations are given by

$$\begin{aligned} \frac{d\alpha_i}{dt} &= -\frac{\partial H_1}{\partial \beta_i} = -\frac{\partial K}{\partial \beta_i} - \frac{\partial K}{\partial r}\frac{\partial \phi}{\partial \beta_i} \\ & \qquad\qquad\qquad (i = 1,\dots,4) \qquad (3.41) \\ \frac{d\beta_i}{dt} &= +\frac{\partial H_1}{\partial \alpha_i} = +\frac{\partial K}{\partial \alpha_i} + \frac{\partial K}{\partial r}\frac{\partial \phi}{\partial \alpha_i} , \end{aligned}$$

where the partial derivatives $\frac{\partial \phi}{\partial \alpha_i}$, $\frac{\partial \phi}{\partial \beta_i}$ $(i = 1,\dots,4)$ can be obtained by solving the equations

$$\frac{\partial\psi}{\partial r}\frac{\partial\phi}{\partial\alpha_i} + \frac{\partial\psi}{\partial\alpha_i} = 0$$

$$(i = 1,\ldots,4)$$

$$\frac{\partial\psi}{\partial r}\frac{\partial\phi}{\partial\beta_i} + \frac{\partial\psi}{\partial\beta_i} = 0$$

with

$$\psi \equiv t - \beta_1 + ru/\alpha_4 + \frac{u}{|u|} k\alpha_4^{-3/2} \sin^{-1}[(k - \alpha_4 r)(k^2 - \alpha_4\alpha_3^2)^{-\frac{1}{2}}]$$

and

$$u = \pm(-\alpha_3^2 + 2kr - \alpha_4 r^2)^{\frac{1}{2}}/r \ .$$

Since the right-hand sides of Equations (3.41) are functions of r, and not t, application of the chain rule to the left-hand sides of Equations (3.41) gives the following result

$$\frac{d\alpha_i}{dr} = -\frac{1}{u}\left[\frac{\partial K}{\partial\beta_i} + \frac{\partial K}{\partial r}\frac{\partial\phi}{\partial\beta_i}\right]$$

$$(i = 1,\ldots,4) \qquad (3.42)$$

$$\frac{d\beta_i}{dr} = \frac{+1}{u}\left[\frac{\partial K}{\partial\alpha_i} + \frac{\partial K}{\partial r}\frac{\partial\phi}{\partial\alpha_i}\right],$$

which only involve α_i's, β_i's, and r.

3.3 A Base Solution in Poincare Variables

As previously noted, the solution of Section(3.2)suffers from a number of defects. Attempts were made to remove the circular singularity by performing a canonical transformation from the $\{\alpha, \beta\}$ - set, defined by Equations (3.40), to some new nonsingular set of variables. These attempts were cumbersome, and even if they had succeeded, the ambiguity of the $\pm$ signs would still be present in the problem. Thus, a different method of attack was used to remove the above mentioned difficulties. The new method described below has the additional advantage of possessing a separable H-J equation.

Consider the Hamiltonian of Equation (3.22). Since a change of independent variable will be performed later, Equation (3.22) will be modified to include time, t, as a coordinate, i.e.,

$$H = [\lambda_1(v^2/r - k/r^2) - \lambda_2 uv/r + \lambda_3 u + \lambda_4 v/r + \lambda_5 \cdot 1] + \frac{T}{m}(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}} , \tag{3.43}$$

where λ_5 is the Lagrange multiplier of the $\dot{t} = 1$ equation.

In Reference 31, a set of orbital parameters, denoted by $\{X, \Lambda\}$, similar to the classic Poincare variables[33] is developed. For elliptical missions these variables are related to the classical Keplerian parameters by

$$\begin{aligned} h &= [a(1 - e^2)/k]^{\frac{1}{2}} \\ q &= e \cos \omega \\ s &= e \sin \omega \\ \theta &= \theta , \end{aligned} \tag{3.44}$$

where a is the semi-major axis, e is the eccentricity, and ω is the argument of perihelion of the base orbit. For the case of circular conditions, ω is taken to be θ_o. For convenience, the parameters defined in Equations (3.44) will be called <u>Poincare variables</u> because of their close relation with the classic Poincare variables.

The coordinate transformations between the polar coordinates and the Poincare variables are

$$
\begin{aligned}
u &= (q \sin\theta - s \cos\theta)/h & \qquad r &= h^2 k/\gamma \\
v &= \gamma/h & \qquad \theta &= \theta
\end{aligned}
\tag{3.45}
$$

and,

$$
\begin{aligned}
h &= rv/k & \qquad q &= (rv^2/k - 1)\cos\theta + (ruv/k)\sin\theta \\
\theta &= \theta & \qquad s &= (rv^2/k - 1)\sin\theta - (ruv/k)\cos\theta \ ,
\end{aligned}
\tag{3.46}
$$

where

$$
\gamma \equiv 1 + q\cos\theta + s\sin\theta \ . \tag{3.47}
$$

Let $X = \{h, q, s, \theta, t\}$ and $\dot{X} = F(X, \alpha)$ be the equations of motion, in the Poincare system, for a vehicle which satisfies the same assumptions as the analysis of Section 3.2. The generalized Hamiltonians in the polar and Poincare systems are

$$
H = \sum_{i=1}^{5} \lambda_i f_i(x, \alpha) \ , \qquad \tilde{K} = \sum_{i=1}^{4} \Lambda_i F_i(X, \alpha) + \Lambda_5^* \ , \tag{3.48}
$$

where Λ_5^* is written instead of Λ_5 since Λ_5^* is an intermediate variable. The equations of motion in the Poincare system need never be developed because of the properties of the two systems defined by the Hamiltonians of Equation (3.48). That is, since h , q , and s are constants of the motion for the Kepler problem, the right-hand sides of their differential equations must be multiplied by thrust. Thus, their conjugate multipliers cannot appear in the coast-arc Hamiltonian, so

$$\tilde{K}_o = \Lambda_4 \dot{\theta} + \Lambda_5^* \dot{t}$$

or,

$$\tilde{K}_o = \Lambda_4 \gamma^2/(h^3 k) + \Lambda_5^* \ , \qquad (3.49)$$

where $\dot{\theta} = v/r = \gamma^2/(h^3 k)$ by Equations (3.18) and (3.45).

The thrusting portion of the Hamiltonian, say $\tilde{K}_1$, can be obtained by substituting for $\lambda_1(X, \Lambda)$ and $\lambda_2(X, \Lambda)$ in the thrust portion of Equation (3.43). This requires the transformation between the two sets of multipliers (which must be derived anyway), and this transformation is defined by the following development.

Since the Hamiltonians in the two systems (see Equation (3.48)) are in the form required by the hypothesis of Theorem (2.3), the Lagrange multiplier transformation is defined by

$$\Lambda_i = \sum_{j=1}^{5} \lambda_j \frac{\partial \phi_j}{\partial X_i} \ , \qquad (i = 1,\ldots,5) \qquad (3.50)$$

where $x = \phi(X)$ is the coordinate transformation defined by Equations (3.45). In expanded form, Equations (3.50) are

$$\begin{aligned}
\Lambda_1 &= -\lambda_1(q \sin\theta - s\cos\theta)/h^2 - \lambda_2\gamma/h^2 + 2\lambda_3 hk/\gamma \\
\Lambda_2 &= (\lambda_1 \sin\theta)/h + (\lambda_2 \cos\theta)/h - (\lambda_3 h^2 k \cos\theta)/\gamma^2 \\
\Lambda_3 &= -(\lambda_1 \cos\theta)/h + (\lambda_2 \sin\theta)/h - (\lambda_3 h^2 k \sin\theta)/\gamma^2 \qquad (3.51) \\
\Lambda_4 &= \lambda_1(\gamma - 1)/h + (\lambda_2/h - \lambda_3 h^2 k/\gamma^2)(s\cos\theta - q\sin\theta) + \lambda_4 \\
\Lambda_5^* &= \lambda_5 \ .
\end{aligned}$$

The inverse transformation is

$$\begin{aligned}
\lambda_1 &= \Lambda_2 h \sin\theta - \Lambda_3 h \cos\theta \\
\lambda_2 &= (h/\gamma)\,\{\Lambda_1 h + \Lambda_2[q + (1+\gamma)\cos\theta] + \Lambda_3[s + (1+\gamma)\sin\theta]\} \\
\lambda_3 &= (\gamma/kh)\,\{\Lambda_1 + \Lambda_2(q + \cos\theta)/h + \Lambda_3(s \mp \sin\theta)/h\} \qquad (3.52) \\
\lambda_4 &= -\Lambda_2 s + \Lambda_3 q + \Lambda_4 \\
\lambda_5 &= \Lambda_5^*
\end{aligned}$$

Therefore, making use of Equations (3.43), (3.49), and (3.52), the total Hamiltonian in Poincare variables is

$$\tilde{K} = \Lambda_4 \gamma^2/(h^3 k) + \Lambda_5^* + h\,\frac{T}{m}\,(A^2 + B^2)^{\frac{1}{2}} \qquad (3.53)$$

where

$$\begin{aligned}
A &\equiv \{\Lambda_1 h + \Lambda_2[q + (1+\gamma)\cos\theta] + \Lambda_3[s + (1+\gamma)\sin\theta]\}/\gamma \\
B &\equiv \Lambda_2 \sin\theta - \Lambda_3 \cos\theta\ .
\end{aligned} \qquad (3.54)$$

Recall from Section (3.2) the implicit relationship difficulties in forming $\{x(\alpha, \beta, t), \lambda(\alpha, \beta, t)\}$. Not only was this inversion impossible, but also the resulting perturbation equations, Equations (3.42), lost their Hamiltonian form. Both of these difficulties are a consequence of time, t, being chosen as the independent variable. Since the solution of this section is to be used in analytic perturbation analyses of the optimal trajectory problem, preservation of the Hamiltonian form is essential. Therefore, the independent variable will be changed from time, t, to the polar angle, θ.

In Section (2.1), a method for changing the independent variable in a conservative Hamiltonian system was given. The system defined by the Hamiltonian of Equation (3.53) satisfies this criteria, i.e.,

$$\tilde{K}(X, \Lambda) \equiv c = \text{constant} , \tag{3.55}$$

so let

$$K^* \equiv (h^3k/\gamma^2) [\tilde{K} - c] . \tag{3.56}$$

Then, using Equations (3.53) and (3.56)

$$K^* = \Lambda_4 + (\Lambda_5^* - c) (h^3k/\gamma^2) + \frac{T}{m} (h^4k/\gamma^2) (A^2 + B^2)^{\frac{1}{2}} . \tag{3.57}$$

Before developing the coast-arc solution, an important point about the development of Equations (3.55), (3.56), and (3.57) must be made. Note that c is the numerical value of the total Hamiltonian $\tilde{K}$. This fact becomes important when an attempt is made to attach physical significance to problems defined by a portion of the total Hamiltonian of Equation (3.57). For example, consider the Hamiltonian for the coast-arc problem with time as the independent variable:

$$\tilde{K}_o = \Lambda_4\gamma^2/(h^3k) + \Lambda_5^* \tag{3.58}$$

Application of the change of independent variable property of Section (2.1) requires

$$K_o^* = (h^3k/\gamma^2) [\tilde{K}_o - c^*] , \tag{3.59}$$

where $\tilde{K}_o(X, \Lambda) \equiv c^*$ and $c^* \neq c$, in general. Then, after substituting Equation (3.58) into Equation (3.59)

$$K_o^* = \Lambda_4 + (\Lambda_5^* - c^*) (h^3k/\gamma^2) . \tag{3.59'}$$

Equation (3.59) is the Hamiltonian for the coast-arc problem with θ as the independent variable. However, note that if $T = 0$ in Equation (3.57), then the resultant sub-Hamiltonian represents the coast-arc problem only if $c = c^*$. Thus, when applying the solution developed below to the coast-arc problem, the value of the constant, c, must be equal to c^*, which is easily determined by evaluating K_o^* at the initial instant of the coast-arc. Other approximations (e.g., circumferential thrust) require a similar analysis.

A simple canonical transformation can be used to form a new Hamiltonian system which does not depend on the constant. Thus, consider

$$\begin{aligned} h &= h, \quad q = q, \quad s = s, \quad \theta = \theta, \quad t = t \\ \lambda_1 &= \lambda_1, \quad \lambda_2 = \lambda_2, \quad \lambda_3 = \lambda_3, \quad \lambda_4 = \lambda_4, \quad \lambda_5 = \lambda_5^* - c . \end{aligned} \tag{3.60}$$

Equations (3.60) represent a canonical transformation since the addition of constants to the canonical variables does not affect the Hamiltonian form. It must be stressed again, though, that the value assigned to c depends upon the physical interpretation given to a sub-Hamiltonian of Equation (3.57).

Upon application of the transformation of Equations (3.60), the total Hamiltonian becomes

$$\begin{aligned} K \equiv K^*[X, \Lambda, \Lambda_5^*(\Lambda_5)] &= \Lambda_4 + \Lambda_5(h^3k/\gamma^2) + \\ &+ \frac{T}{m}(h^4k/\gamma^2)(A^2 + B^2)^{\frac{1}{2}} . \end{aligned} \tag{3.61}$$

The nonthrust portion of K is then

$$K_o \equiv \Lambda_4 + \Lambda_5(h^3k/\gamma^2). \tag{3.62}$$

Since the solution of the H-J equation for the Hamiltonian of Equation (3.62) is to be used as a base solution in canonical perturbation analyses, it will be more convenient to consider the solution of the modified H-J equation

$$K_o(X, \frac{\partial S}{\partial X}) = \alpha_1 . \tag{3.63}$$

The only difference between the solution of Equation (3.63) and the equation

$$\frac{\partial S}{\partial \theta} + K_o = 0 \tag{3.64}$$

is that: (i) the solution of Equation (3.63) (which does not contain the independent variable, θ) will contain nine canonic constants and a nonzero Hamiltonian remainder, α_1 , which involves a simple quadrature for the remaining canonic constant; whereas (ii) the solution of Equation (3.64) contains the independent variable θ , which is not desirable for future perturbation studies.

Before attacking Equation (3.63), a simple canonical transformation, which takes advantage of the fact that neither Λ_1 , Λ_2 , nor Λ_3 appear explicitly in the Hamiltonian, will be performed. The new momenta are defined to be

$$p_1 = h , \quad p_2 = q , \quad p_3 = s , \quad p_4 = \Lambda_4 , \quad p_5 = \Lambda_5 . \tag{3.65}$$

Then by Example (2.3), the new coordinates are

$$q_1 = -\Lambda_1 , \quad q_2 = -\Lambda_2 , \quad q_3 = -\Lambda_3 , \quad q_4 = \theta , \quad q_5 = t . \tag{3.66}$$

Equation (3.63) can then be written as

$$\frac{\partial S}{\partial q_4} + k\left(\frac{\partial S}{\partial q_1}\right)^3 \frac{\partial S}{\partial q_5} \left[1 + \frac{\partial S}{\partial q_2} \cos q_4 + \frac{\partial S}{\partial q_3} \sin q_4\right]^{-2} = \alpha_1 , \tag{3.67}$$

which can be solved by separation of variables. That is, assume

$$S = S_1(q_1) + S_2(q_2) + S_3(q_3) + S_4(q_4) + S_5(q_5) . \tag{3.68}$$

Then, after substituting Equation (3.68) into Equation (3.67) and applying the usual separation of variables independence arguments, the following constant relations are formed

$$\begin{aligned} \frac{\partial S}{\partial q_1} &= \alpha_2(=h) & \frac{\partial S}{\partial q_3} &= \alpha_4(=s) \\ \frac{\partial S}{\partial q_2} &= \alpha_3(=q) & \frac{\partial S}{\partial q_5} &= \alpha_5(=\Lambda_5) . \end{aligned} \tag{3.69}$$

Thus,

$$\begin{aligned} S = {} & \alpha_2 q_1 + \alpha_3 q_2 + \alpha_4 q_3 + \alpha_5 q_5 + \alpha_1 q_4 \\ & - \alpha_2^3 \alpha_5 k \int [1 + \alpha_3 \cos q_4 + \alpha_4 \sin q_4]^{-2} dq_4 . \end{aligned} \tag{3.70}$$

The indefinite integral of Equation (3.70) can be evaluated by making the substitution $z \equiv \tan \frac{q_4}{2}$, with the result

$$\begin{aligned} \int \gamma^{-2} dq_4 = {} & [(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 + \alpha_4(1 + \cos q_4)] \\ & \cdot [\gamma(1 - \alpha_3)(1 - \alpha_3^2 - \alpha_4^2)]^{-1} + 2(1 - \alpha_3^2 - \alpha_4^2)^{-3/2} \\ & \cdot \tan^{-1} \{[(1 - \alpha_3)\tan(q_4/2) + \alpha_4](1 - \alpha_3^2 - \alpha_4^2)^{-\frac{1}{2}}\} . \end{aligned} \tag{3.71}$$

The evaluation of Equation (3.71) requires $(1 - e^2) > 0$, so the result is only valid for circular and elliptical conditions. In Section (3.4), the hyperbolic and parabolic cases are discussed.

The set $\{\alpha_1, \ldots, \alpha_5\}$, defined by Equations (3.67) and (3.69), gives half of the constants of the motion for the coast-arc problem. Four of the five remaining constants can be formed by applying Jacobi's Theorem, i.e., $\beta_i \equiv \frac{\partial S}{\partial \alpha_i}$. Then, the remaining constants are defined by the β_2 , β_3 , β_4 , and β_5 equations and a quadrature involving the β_1 equation with

$$
\begin{aligned}
\beta_1 &= q_4 \\
\beta_2 &= q_1 - 3\alpha_2^2\alpha_5 k[(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 + \alpha_4(1 + \cos q_4)]Y_1 \\
&\quad - 6\alpha_2^2\alpha_5 k\, Y_2(1 - \alpha_3^2 - \alpha_4^2)^{-3/2} \\
\beta_3 &= q_2 - \alpha_2^2\alpha_5 k\{[\alpha_3\alpha_4(1 + \cos q_4) + (\alpha_3 + \alpha_4^2 - 1)\sin q_4] \\
&\quad \cdot (1 - \alpha_3^2 - \alpha_4^2)^{-1}\, Y_1 + (2\alpha_3 - 1)Y_1 \sin q_4 \\
&\quad + 6\alpha_3(1 - \alpha_3^2 - \alpha_4^2)^{-5/2}\, Y_2 + Y_1^2[(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 \qquad (3.72) \\
&\quad + \alpha_4(1 + \cos q_4)] \cdot [\gamma(1 - \alpha_3^2 - \alpha_4^2) + (1 - \alpha_3)(2\alpha_3\gamma - (1-\alpha_3^2-\alpha_4^2)\cos q_4)]\} \\
\beta_4 &= q_3 - \alpha_2^3\alpha_5 k\, \{Y_1(1 - \alpha_3^2 - \alpha_4^2)^{-1}[(2 - 2\alpha_3^2 - \alpha_4^2)\,(1 + \cos q_4) \\
&\quad + \alpha_4(3 - \alpha_3 - 2\alpha_3^2 - 2\alpha_4^2)\sin q_4] + 6\alpha_4(1 - \alpha_3^2 - \alpha_4^2)^{-5/2}\, Y_2 \\
&\quad + Y_1^2\,(1 - \alpha_3)\,[2\alpha_4\gamma - (1 - \alpha_3^2 - \alpha_4^2)\sin q_4]\,[(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 \\
&\quad + \alpha_4(1 + \cos q_4)]\} \\
\beta_5 &= q_5 - \alpha_2^3 k\, \{Y_1[(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 + \alpha_4(1 + \cos q_4)] \\
&\quad + 2Y_2(1 - \alpha_3^2 - \alpha_4^2)^{-3/2}\} ,
\end{aligned}
$$

where

$$Y_1 \equiv [\gamma(1 - \alpha_3)(1 - \alpha_3^2 - \alpha_4^2)]^{-1}$$

$$Y_2 \equiv \tan^{-1}\{[(1 - \alpha_3)\tan(q_4/2) + \alpha_4](1 - \alpha_3^2 - \alpha_4^2)^{-\frac{1}{2}}\} \qquad (3.73)$$

$$\gamma \equiv 1 + \alpha_3 \cos q_4 + \alpha_4 \sin q_4$$

Equations (3.63), (3.69), and (3.72) are easily inverted to give $\{q(\alpha, \beta), p(\alpha, \beta)\}$, and this represents the solution to the coast-arc problem.

The β_i -relationships of Equation (3.72) are somewhat more complicated than the comparable relations of Equations (3.40). However, they possess only the zero-energy singularity (i.e., $1 - \alpha_3^2 - \alpha_4^2 \equiv 0$ at zero-energy) and they do not contain the $\pm$ sign ambiguity. Furthermore, since $\alpha_3 = e \cos \omega$ and $\alpha_4 = e \sin \omega$, the expressions are especially useful for near-circular conditions if they are expanded about $(\alpha_3 = 0, \alpha_4 = 0)$, i.e., zero-eccentricity. With the zero-eccentricity assumption, Equations (3.72) become

$$\begin{aligned} \beta_1 &= q_4 \\ \beta_2 &= q_1 - 3\alpha_2^2\alpha_5 k q_4 \\ \beta_3 &= q_2 + 2\alpha_2^3\alpha_5 k \sin q_4 \\ \beta_4 &= q_3 - 2\alpha_2^3\alpha_5 k(1 + \cos q_4) \\ \beta_5 &= q_5 - \alpha_2^3 k q_4 . \end{aligned} \qquad (3.74)$$

In Chapter 4, Equations (3.74) prove to be useful in analyzing the circumferential thrust problem.

An inconvenient quality of Equations (3.72) is the occurrence of Y_2. That is, when a multirevolution trajectory is considered, one must "keep track" of the arctangent function. However, since α_3 and α_4 are less than one for elliptical trajectories, Y_2 can be expanded in a Taylor series about $(\alpha_3 = 0 , \alpha_4 = 0)$. This expansion is rapidly convergent for near-circular missions and the troublesome bookkeeping task is avoided.

The expansion for Y_2 , to second order in eccentricity, is

$$Y_2 \simeq \frac{1}{2} [q_4 - \alpha_3 \sin q_4 + \alpha_4(1 + \cos q_4) + \\ + \alpha_3\alpha_4 \sin^2 q_4 + \frac{1}{2} (\alpha_3^2 - \alpha_4^2) \sin q_4 \cos q_4]. \tag{3.75}$$

Equation (3.75) is extremely useful in the study of multirevolution optimal low-thrust trajectories. For example, on a representative optimal escape trajectory with circular initial conditions ($F/W_o = 5 \times 10^{-4}$, 70.5 revolutions, and travel time $\simeq$ 1.28 million seconds), the approximation of Equation (3.75) held at least six digit accuracy for the first 85% of the trajectory, and at least three digits for the remainder of the trajectory. (On representative multirevolution circular orbit transfers, the approximation held six digits for the entire trajectory.) The reason for this convergence behavior is indicated by Figures 2 and 3. That is, for the major initial portion of both classes of trajectories, the eccentricity is less than 0.01. Thus, Figure 3, which represents a relatively small range angle, shows that the approximation holds six digits because of the small eccentricity. By the time $e > 0.01$, the range angle has reached a relatively large angle. Thus, Figure 4, which represents a relatively large range angle, shows that the approximation still holds six digits since the increase in eccentricity is compensated for by the large range angle. However, with respect to the final portion of the escape trajectory, the expansion tends to become invalid as $q^2 + s^2 = e^2 \to 1$ (the escape condition).

3.4 The Hyperbolic and Parabolic Coast-Arc Solutions

In this section, the generating functions for the hyperbolic and parabolic coast-arcs are developed. The total solution, for each of these cases, can be formed easily then by evaluating the partial derivatives of the generating function with respect to each of the α_i's . The same set of α_i's is valid for each of three cases. Only the β_i's change.

The evaluation of Equation (3.71) required $(1 - e^2) > 0$. The evaluations where $(1 - e^2) = 0$ (parabolic) and $(1 - e^2) < 0$ (hyperbolic) will now be developed. After applying the substitution $z \equiv \tan q_4/2$ and performing a sequence of trigonometric manipulations, the integral can be expressed as

$$\int \gamma^{-2} dq_{\_} = 2 \left[\int (az^2 + bz + c)^{-2} dz + \int z^2 (az^2 + bz + c)^{-2} dz \right], \qquad (3.76)$$

where $a \equiv 1 - \alpha_3$, $b \equiv 2\alpha_4$, and $c \equiv 1 + \alpha_3$. The three cases are defined by the discriminant

$$b^2 - 4ac = -4[1 - \alpha_3^2 - \alpha_4^2] = -4(1 - e^2).$$

First, the parabolic case will be considered. In this case, $b^2 - 4ac = 0$, so

$$z^2 + \frac{bz}{a} + \frac{c}{a} = [z + \alpha_4/(1 - \alpha_3)]^2 .$$

Then, the component integrals of Equation (3.76) can be readily determined, and substitution of these results into Equation (3.70) gives

$$
\begin{aligned}
S_p \equiv\ & \alpha_2 q_1 + \alpha_3 q_2 + \alpha_4 q_3 + \alpha_5 q_5 + \alpha_1 q_4 \\
& + 2\,\alpha_2^3 \alpha_5 k \left[\alpha_4 + (1 - \alpha_3)\tan \frac{q_4}{2}\right]^{-2} \left\{\tan \frac{q_4}{2}\right. \\
& \left. + \frac{1}{3}\left[\alpha_4 + (1 - \alpha_3)\tan \frac{q_4}{2}\right]^{-1} \left[1 - \alpha_3 + \alpha_4^2/(1 - \alpha_3)\right]\right\}
\end{aligned}
\tag{3.77}
$$

The set $\{\alpha,\ \beta \equiv \frac{\partial S_p}{\partial \alpha}\}$ defines the parabolic solution.

Finally, the hyperbolic case will be considered. With $b^2 - 4ac > 0$, the component integrals of Equation (3.76) are easily determined by reference to a standard tables of integrals. Substitution of these evaluations in Equation (3.70) gives the hyperbolic generating function

$$
\begin{aligned}
S_H \equiv\ & \alpha_2 q_1 + \alpha_3 q_2 + \alpha_4 q_3 + \alpha_5 q_5 + \alpha_1 q_4 - \alpha_2^3 \alpha_5 k \\
& \cdot \left[\{(\alpha_3^2 + \alpha_4^2 - \alpha_3)\sin q_4 + \alpha_4(1 + \cos q_4)\} \cdot \{\gamma(1 - \alpha_3)(1 - \alpha_3^2 - \alpha_4^2)\}^{-1}\right. \\
& \left. + 2(\alpha_3^2 + \alpha_4^2 - 1)^{-3/2} \tanh^{-1} \{[(1 - \alpha_3)\tan \frac{q_4}{2} + \alpha_4](\alpha_3^2 + \alpha_4^2 - 1)^{-\frac{1}{2}}\}\right] .
\end{aligned}
\tag{3.78}
$$

Note that the essential difference between the elliptic and hyperbolic solutions is the occurrence of the arctangent and archyperbolictangent functions, respectively. This analogy could possibly be used to define a set of universal variables for the problem.

3.5 Perturbation Attempts for Feedback Guidance Functions

The solution of Section (3.3) reduces the original Hamiltonian (Equation (3.61)) to

$$K = \alpha_1 + \{T\,\alpha_2^4 k/[m(\alpha, \beta)\gamma^2(\alpha, \beta)]\}\cdot \cdot [A^2(\alpha, \beta) + B^2(\alpha, \beta)]^{\frac{1}{2}}, \tag{3.79}$$

where the explicit forms of the unspecified functions (of α_i's and β_i's) are given by Equations (3.54), (3.69), and (3.72). The ideal second step in the process would be the formation of an approximation to the Hamiltonian system defined by Equation (3.79) which includes the effects of <u>both</u> $A(\alpha, \beta)$ and $B(\alpha, \beta)$. In the next chapter, approximations which incorporate the A-term are presented. As will be shown, these approximations do not allow for feedback guidance since they define circumferential thrust solutions. To obtain feedback effects, both A and B (or approximations of these terms) must appear in the Hamiltonian.

Analytic attempts to incorporate the B-term into the solution were unsuccessful. Since research in this area should be continued, some of the reasons for failure are presented here for future investigators.

First of all, consider the functional dependence of the A-, B-, and K-terms:

$$\begin{aligned} A &= A(\beta_1, \beta_2, \beta_3, \beta_4, \_, \_, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ B &= B(\beta_1, \_, \beta_3, \beta_4, \_, \_, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ K &= K(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \end{aligned} \tag{3.80}$$

If the canonical perturbation equations are considered (i.e., $\alpha_i' = -\frac{\partial K}{\partial \beta_i}$, $\beta_i' = \frac{\partial K}{\partial \alpha_i}$ where $(\ )' \equiv \frac{d(\ )}{d\,\theta}$), then

$$\begin{aligned} \alpha_i' &= \frac{(\ldots)}{\sqrt{A^2 + B^2}} + (\ldots) \qquad (i = 1, 2, 3, 4) \\ \beta_i' &= \frac{(\ldots)}{\sqrt{A^2 + B^2}} + (\ldots). \qquad (i = 2, 3, 4, 5) \end{aligned} \tag{3.81}$$

The term $(A^2 + B^2)$ is very complicated and since it occurs in a radical in the denominators of the perturbationequations, the situation is complicated even further. Thus, approximations are necessary.

Before considering approximations of the $\sqrt{A^2 + B^2}$ term, another approximation should be mentioned. For most low-thrust missions with

$$m = m_o + \dot{m}_o(t - t_o) ,$$

the term $\frac{\dot{m}_o}{m_o}$ is very small and $m_o > \dot{m}_o(t - t_o)$. Thus, a binomial expansion for $m(t)$ is applicable

$$\frac{1}{m} = \frac{1}{m_o} [1 - \frac{\dot{m}_o}{m_o} t + - \ldots] ,$$

and good results can even be obtained by assuming

$$\frac{1}{m} \simeq \frac{1}{m_o} .$$

The constant mass assumption results in the vanishing of β_5 from $K(\alpha, \beta)$, so α_5 becomes a constant of the motion for this total problem approximation.

The optimal control is given by

$$\tan \alpha = \frac{B}{A} .$$

For many multirevolution optimal low-thrust missions with circular initial conditions, the absolute value of the control angle is less than 45°. Thus,

$$\left|\frac{B}{A}\right| < 1 \quad \rightarrow \quad |B| < |A| .$$

In this case,

$$\sqrt{A^2 + B^2} \simeq A[1 + \frac{1}{2}(\frac{B}{A})^2 - + \ldots] .$$

Therefore, to incorporate the effects of the B-term, one must consider at least

$$\sqrt{A^2 + B^2} \simeq A + \frac{1}{2}\frac{B^2}{A}$$

But, even with this simple form the occurrence of A in the denominator presents an effect similar to the radical in the denominator of Equations (3.81).

For example, suppose that the instantaneous conditions are always near circular. Then,

$$A + \frac{1}{2}\frac{B^2}{A} \simeq [A + \frac{1}{2}\frac{B^2}{A}]_{\alpha_3=\alpha_4=0} ,$$

where, recalling that $\beta_1 = q_4 = \theta$,

$$A|_{\alpha_3=\alpha_4=0} = -\alpha_2\beta_2 - 2\beta_3\cos\beta_1 - 2\beta_4\sin\beta_1 - \alpha_2^3\alpha_5 k(3\beta_1 + 4\sin\beta_1)$$

$$B|_{\alpha_3=\alpha_4=0} = -\beta_3\sin\beta_1 + \beta_4\cos\beta_1 + 2\alpha_2^3\alpha_5 k(1 + \cos\beta_1) .$$

The approximate Hamiltonian for this case, along with the constant mass assumption and $\gamma \simeq \gamma|_{\alpha_3=\alpha_4=0}$, is

$$K = \alpha_1 + \frac{T}{m_o}\alpha_2^4 k\{-\alpha_2\beta_2 - 2\beta_3\cos\beta_1 - 2\beta_4\sin\beta_1 - \alpha_2^3\alpha_5 k(3\beta_1 + 4\sin\beta_1) + + \frac{1}{2}\frac{[-\beta_3\sin\beta_1 + \beta_4\cos\beta_1 + 2\alpha_2^3\alpha_5 k(1 + \cos\beta_1)]^2}{[-\alpha_2\beta_2 - 2\beta_3\cos\beta_1 - 2\beta_4\sin\beta_1 - \alpha_2^3\alpha_5 k(3\beta_1 + 4\sin\beta_1)]}\} \quad (3.82)$$

The canonical perturbation equations for α_1, α_2, α_3, α_4, β_2, and β_5 will all contain the term $\frac{1}{A^2}$, which is similar to the denominator problem of Equations (3.81). Another inconvenient property of the Hamiltonian of Equation (3.82) is that it is not a periodic function of β_1(i.e., θ).

In the next chapter, a solution of the H-J equation for the Hamiltonian of Equation (3.82) is obtained for the case $B = 0$. It appears that future studies which attempt to incorporate the B-term into the analysis will have to thoroughly investigate the possibility of simplifying the $\frac{1}{A}$ -expression.

CHAPTER 4

HAMILTONIAN CIRCUMFERENTIAL THRUST ANALYSIS

In 1953, Tsien[44] used asymptotic expansions to obtain approximate solutions of the state differential equations for a low-thrust spiral trajectory. His solutions showed that a circumferential thrust program is preferable to a radial thrust program in an escape mission. Since that time, considerable research has been expended on the low-thrust problem. Pitkin[32], in 1966, published a bibliography of nearly 250 papers concerned with the low-thrust trajectory problem. Many of these papers duplicate previously given results, and not many improve the solutions obtained by Tsien.

The numerical studies of Irving[16], Moeckel[27], Melbourne[24], and Hinz[15] demonstrate that up to escape conditions, circumferential and tangential thrust programs give near optimal results for multi revolution trajectories. The general character of optimal low-thrust escape trajectories, with circular initial conditions, is the following: the vehicle stays very near circular conditions for the major portion of the trajectory as it gains energy by increasing the radius and decreasing the speed. Then, after a considerable increase in radial distance, the gravitational force becomes less dominant (with respect to the thrust force) and in the last revolution or two, the vehicle goes from a near-circular condition to the escape condition. Thus, for the major portion of the trajectory, near-circular conditions are maintained. The character of optimal circular orbit transfer trajectories is similar to the optimal escape in that its instantaneous conditions are also near-circular. In Chapter 7 these trends are validated numerically.

The primary objective of this chapter is not the development of a new approximate circumferential thrust solution (although this is accomplished), but the development of the corresponding Lagrange multiplier solutions and a Hamiltonian formulation of known results. All of the previous solutions are concerned only with the state variable solutions since, in a physical sense, the multipliers have no meaning when the control is specified. However, from the differential equation point of view, the circumferential multiplier solutions may approximate the true optimal multipliers, if the optimal control is near-circumferential. Thus, these solutions may be of use in an open-loop guidance scheme or in mission planning studies.

A Hamiltonian formulation is desirable for a number of reasons: (1) once a portion of the problem is solved, the solution to a more general problem can be attacked by adding terms to the Hamiltonian function instead of attacking the full problem again with the new effect added; (2) the classical methods of Hamiltonian perturbation theory are then applicable to the problem; and (3) the analytic solution of the Hamilton-Jacobi equation is sometimes cleaner and easier than the straight-forward integration of Hamilton's equations for problems with a relatively large number of variables (such as the optimal trajectory problem).

4.1 The Hamilton-Jacobi Solution of Previous Circumferential Thrust Approximations

A representative collection of approximate circumferential thrust solutions are those of Tsien[44], Melbourne[24], Perkins[30], Hinz[15], and Johnson and Stumpf[18]. Recently Shi and Eckstein[39] applied the two-variable expansion method to the problem. All of the solutions mentioned above assume a constant low-thrust acceleration (i.e., $F/m \equiv F/m_o$) and the solutions presented below are also for constant acceleration since the mass-flow rate of a low-thrust engine is very small.

The total generalized Hamiltonian for a planar representation of an optimal continuous-thrust trajectory problem is given by Equation (3.61):

$$K = \Lambda_4 + h^3 k \Lambda_5/\gamma^2 + (T/m)\,(h^4 k/\gamma^2)\,\sqrt{A^2 + B^2}\ . \tag{4.1}$$

If $A >> B$ and $m = m_o + \dot{m}_o t$ (assuming $t_o = 0$), then Equation (4.1) can be written as

$$\begin{aligned} K = \ & \Lambda_4 + h^3 k \Lambda_5/\gamma^2 + (T/m_o)\,(1 - \delta t + \delta^2 t^2 + \ldots)\cdot \\ & \cdot\ (h^4 k/\gamma^2)\, A\{1 + \tfrac{1}{2}\,(B/A)^2 + \ldots\}\ , \end{aligned} \tag{4.2}$$

where $\delta \equiv \dot{m}_o/m_o$. The expansion for mass is in a convenient form for future studies which take variable mass into account. It is easily verified that the state differential equations for circumferential thrust are given by the partial derivatives, with respect to the multipliers, of the Hamiltonian:

$$K = \Lambda_4 + h^3 k \Lambda_5/\gamma^2 + (T/m)\,(h^4 k/\gamma^2) A\ , \tag{4.3}$$

i.e., $B = 0$ (recall $\sin\alpha = B(A^2 + B^2)^{-\frac{1}{2}}$ and $\alpha \equiv 0$ for circumferential thrust).

The analyses of the investigators mentioned above were performed in polar coordinates, whereas the Hamiltonian of Equation (4.3) is in Poincare variables. The Poincare formulation gives a clearer insight into the main simplifying assumption that the instantaneous conditions are near-circular (i.e., the eccentricity is near-zero). Since $q \equiv e\cos\omega$ and $s \equiv e\sin\omega$, it follows that q and s are near zero. For this case, the complete Hamiltonian of Equation (4.3) can be written as:

$$K = \Lambda_4 + h^3 k\Lambda_5 + (h^4 k\ T/m_o)\ (1 - \delta t + \delta^2 t^2 + \dots)\cdot$$
$$\cdot\ (\Lambda_1 h + 2\Lambda_2 \cos\theta + 2\Lambda_3 \sin\theta) + O(q, s)\ , \tag{4.4}$$

where the expansions for γ^{-1} and γ^{-2} are given in Appendix B. Then, after making the small eccentricity and mass-flow rate assumptions, the approximate circumferential Hamiltonian is

$$K = \Lambda_4 + h^3 k\Lambda_5 + (h^4 k\ T/m_o)\ (\Lambda_1 h + 2\Lambda_2 \cos\theta + 2\Lambda_3 \sin\theta)\ . \tag{4.5}$$

A simple canonical transformation which changes h to a momenta and Λ_1 to a coordinate facilitates the integration of the H-J equation, i.e.,

$$\begin{aligned} p_1 &= h\ , & p_2 &= \Lambda_2\ , & p_3 &= \Lambda_3\ , & p_4 &= \Lambda_4\ , & p_5 &= \Lambda_5 \\ q_1 &= -\Lambda_1\ , & q_2 &= q\ , & q_3 &= s\ , & q_4 &= \theta\ , & q_5 &= t\ . \end{aligned} \tag{4.6}$$

Then, with θ as the independent variable, the H-J equation for the Hamiltonian of Equation (4.5) is

$$\frac{\partial S}{\partial\theta} + \frac{\partial S}{\partial q_4} + k\left(\frac{\partial S}{\partial q_1}\right)^3 \frac{\partial S}{\partial q_5} + \frac{T}{m_o} k \left(\frac{\partial S}{\partial q_1}\right)^4 \cdot$$
$$\cdot\ \left(-q_1 \frac{\partial S}{\partial q_1} + 2\frac{\partial S}{\partial q_2}\cos q_4 + 2\frac{\partial S}{\partial q_3}\sin q_4\right) = 0\ . \tag{4.7}$$

Since neither θ , q_2 , q_3 , nor q_5 appear explicitly in Equation (4.7), a partial separation of variables is possible, i.e., assume

$$S = S_o(\theta) + S_1(q_2) + S_2(q_3) + S_3(q_5) + S^*(q_1, q_4)\ . \tag{4.8}$$

By substituting Equation (4.8) into Equation (4.7) and applying the usual independence arguments, four constants, of the five necessary for a complete solution, are obtained

$$\frac{\partial S}{\partial \theta} = a_1 , \quad \frac{\partial S}{\partial q_2} = a_2 , \quad \frac{\partial S}{\partial q_3} = a_3 , \quad \frac{\partial S}{\partial q_5} = a_4 . \tag{4.9}$$

The H-J equation is now reduced to an equation with only two independent variables

$$\begin{aligned} \psi(q_1, q_4) \equiv\ & a_1 + \frac{\partial S^*}{\partial q_4} + a_4 k \left(\frac{\partial S^*}{\partial q_1}\right)^3 + (k_T/m_o) \left(\frac{\partial S^*}{\partial q_1}\right)^4 \cdot \\ & \cdot \left(-q_1 \frac{\partial S^*}{\partial q_1} + 2a_2 \cos q_4 + 2a_3 \sin q_4\right) = 0 \end{aligned} \tag{4.10}$$

The remaining constant of the complete solution can be obtained by considering the characteristic system for Equation (4.10). In particular

$$\begin{aligned} \frac{d\left(\frac{\partial S^*}{\partial q_1}\right)}{d\tau} &= - \frac{\partial \psi}{\partial q_1} = (k_T/m_o) \left(\frac{\partial S^*}{\partial q_1}\right)^5 \\ \frac{dq_4}{d\tau} &= \frac{\partial \psi}{\partial\left(\frac{\partial S^*}{\partial q_4}\right)} = 1 . \end{aligned} \tag{4.11}$$

Equations (4.11) can be written as the single equation

$$\frac{d\left(\frac{\partial S^*}{\partial q_1}\right)}{dq_4} = (k_T/m_o) \left(\frac{\partial S^*}{\partial q_1}\right)^5 ,$$

or,

$$\int \left(\frac{\partial S^*}{\partial q_1}\right)^{-5} d\left(\frac{\partial S^*}{\partial q_1}\right) = (kT/m_o) \int dq_4 + \text{constant} .$$

Upon integration

$$-\left(\frac{\partial S^*}{\partial q_1}\right)^{-4}/4 = (kT/m_o)q_4 - a_5^{-4}/4 , \tag{4.12}$$

where the functional form of a_5 is chosen in such a way that it reduces to the initial value of h. Equation (4.12) can be written more conveniently as

$$\frac{\partial S^*}{\partial q_1} = a_5(1 - \varepsilon q_4)^{-\frac{1}{4}} , \tag{4.13}$$

where

$$\varepsilon \equiv 4(kT/m_o)a_5^4 . \tag{4.14}$$

The development of the generating function S now only requires the integration of the Pfaffian equation

$$dS^* = \frac{\partial S^*}{\partial q_1} dq_1 + \frac{\partial S^*}{\partial q_4} dq_4 , \tag{4.15}$$

where $\frac{\partial S^*}{\partial q_4}$ can be determined by substituting Equation (4.13) into Equation (4.10):

$$\begin{aligned}\frac{\partial S^*}{\partial q_4} = & -a_1 - a_4ka_5^3(1 - \varepsilon q_4)^{-3/4} + \\ & - (kT/m_o)a_5^4 (1 - \varepsilon q_4)^{-1} [-q_1a_5(1 - \varepsilon q_4)^{-\frac{1}{4}} + 2a_2\cos q_4 \\ & + 2a_3\sin q_4] .\end{aligned} \tag{4.16}$$

Since the functional dependence of Equation (4.15) is

$$dS^* = \frac{\partial S^*}{\partial q_1}(q_4)\, dq_1 + \frac{\partial S^*}{\partial q_4}(q_1, q_4) dq_4 \ ,$$

it follows by the arguments of pp. 67-68 of Reference 36 that S^* is necessarily of the form

$$S^* = \frac{\partial S^*}{\partial q_1} q_1 + f(q_4) \ . \tag{4.17}$$

The function $f(q_4)$ is easily obtained by integrating

$$\int \left(\frac{\partial S^*}{\partial q_4}\right)' dq_4 \ ,$$

where $\left(\frac{\partial S^*}{\partial q_4}\right)'$ is that portion of $\frac{\partial S^*}{\partial q_4}$ which does not contain q_1 . Thus,

$$\begin{aligned} f(q_4) &= a_1 q_4 + \text{[illegible]} \\ &- \frac{a_2 \varepsilon}{2} \int \frac{\cos q_4}{(1-\varepsilon q_4)}\, dq_4 - \frac{a_3 \varepsilon}{2} \int \frac{\sin q_4}{(1-\varepsilon q_4)}\, dq_4 \ . \end{aligned} \tag{4.18}$$

The indefinite integrals in Equation (4.18) are not integrable in closed-form. However, as will be shown in Chapter 7, $|\varepsilon q_4| < 1$ for a large class of circular orbit transfers and escape trajectories. For example, on an optimal circular orbit transfer from an altitude of 200 statute miles to an altitude of 22,300 statute miles (corresponding to the altitude of a twenty-four hour satellite) with $T/W_o = 5\text{x}10^{-4}$ and involving 68.3 revolutions about the earth, the maximum value of $|\varepsilon q_4|$ was less than 0.95. Thus, $(1-\varepsilon q_4)^{-1}$ can be expanded by means of a binomial series. Actually this is just a Taylor series expansion about $\varepsilon q_4 = 0$. Since εq_4 reached a maximum value of approximately 0.95 on many of the missions studied, $(1 - \varepsilon q_4)^{-1}$ is most effectively expanded in a Taylor series about $\epsilon q_4 = C^*$, where C^* is to be specified. For transfers of only a few revolutions, $C^* = 0$ (i.e., the binomial series) should give sufficient accuracy. For transfers of many revolutions, $C^* = 0.4$ is a convenient value.

Define $C \equiv 1 - C^*$. Then, the Taylor series is

$$
\begin{aligned}
(1 - \epsilon q_4)^{-1} &= 1./C + (\epsilon q_4 - C^*)/C^2 + \ldots \\
&\ldots + (\epsilon q_4 - C^*)^n/[(n-1)!C^{n+1}] + \ldots \quad .
\end{aligned}
\tag{4.19}
$$

The indefinite integrals of Equation (4.18) can now be evaluated in series form to give

$$
\begin{aligned}
f(q_4) &= -a_1 q_4 + a_4(1 - \epsilon q_4)^{1/4}/(a_5 T/m_o) \\
&- \frac{a_2}{2}\{\epsilon N_o \sin q_4 + \epsilon^2 N_1(\cos q_4 + q_4 \sin q_4) \\
&+ \epsilon^3 N_2[2q_4 \cos q_4 + (q_4^2 - 2)\sin q_4] \\
&+ \epsilon^4 N_3[(3q_4^2 - 6)\cos q_4 + (q_4^3 - 6q_4)\sin q_4] + \ldots\} \\
&- \frac{a_3}{2}\{-\epsilon N_o \cos q_4 + \epsilon^2 N_1(\sin q_4 - q_4 \cos q_4) + \\
&+ \epsilon^3 N_2[2q_4 \sin q_4 + (2 - q_4^2)\cos q_4] + \\
&+ \epsilon^4 N_3[(3q_4^2 - 6)\sin q_4 + (6q_4 - q_4^3)\cos q_4] + \ldots\} \ ,
\end{aligned}
\tag{4.20}
$$

where

$$
\begin{aligned}
N_o &\equiv \frac{1}{C}\left[1 - \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(\frac{-C^*}{C}\right)^n\right] \\
N_1 &\equiv \frac{1}{C^2}\left[1 - \sum_{n=1}^{\infty} \frac{n}{(n-1)!}\left(\frac{-C^*}{C}\right)^n\right] \\
N_2 &\equiv \frac{1}{C^3}\left[1 - \frac{3}{2}\left(\frac{C^*}{C}\right) + \left(\frac{C^*}{C}\right)^2 - + \ldots\right]
\end{aligned}
\tag{4.21}
$$

$$N_3 \equiv \frac{1}{c^4}[1 - \frac{2}{3}(\frac{c^*}{c}) + \frac{5}{12}(\frac{c^*}{c})^2 - + \ldots] .$$

The N_i - expressions are determined by the Taylor series expansion of $(1 - \varepsilon q_4)^{-1}$, with N_i representing the coefficient of $(\varepsilon q_4)^i$.

By combining Equations (4.8), (4.19), (4.17), and (4.20), the generating function is determined

$$S = a_1\theta + a_2q_2 + a_3q_3 + a_4q_5 + q_1a_5(1 - \varepsilon q_4)^{-\frac{1}{4}} + f(q_4) . \tag{4.22}$$

Recalling that ε depends upon a_5 , the remaining canonic constants can be determined by applying Jacobi's theorem (i.e., $b_i \equiv \frac{\partial S}{\partial a_i} =$ constant):

$$\begin{aligned}
b_1 &= \theta - q_4 \\
b_2 &= q_2 - \frac{\varepsilon}{2}\{N_o\sin q_4 + \varepsilon N_1(\cos q_4 + q_4\sin q_4) + \ldots\} \\
b_3 &= q_3 - \frac{\varepsilon}{2}\{-N_o\cos q_4 + \varepsilon N_1(\sin q_4 - q_4\cos q_4) + \ldots\} \\
b_4 &= q_5 + (1 - \varepsilon q_4)^{\frac{1}{4}}/(a_5T/m_o) \\
b_5 &= q_1(1 - \varepsilon q_4)^{-5/4} - a_4(1 - \varepsilon q_4)^{-3/4}/(a_5^2T/m_o) \\
&- \frac{2a_2}{a_5}\{\varepsilon N_o\sin q_4 + 2\varepsilon^2N_1(\ldots) + \\
&+ 3\varepsilon^3N_2[\ldots] + 4\varepsilon^4N_3[\ldots] + \ldots\} \\
&- \frac{2a_3}{a_5}\{-\varepsilon N_o\cos q_4 + 2\varepsilon^2N_1(\ldots) \\
&+ 3\varepsilon^3N_2[\ldots] + 4\varepsilon^4N_3[\ldots] + \ldots\} ,
\end{aligned} \tag{4.23}$$

where (...) and [...] represent the bracketed terms in Equation (4.20) which are multiplied by a_2N_i and a_3N_i $(i = 1, 2, \ldots)$. It should also be noted that the expression $\frac{\partial\varepsilon}{\partial a_5} = 4\varepsilon/a_5$ was used in developing b_5 .

The series expansions for b_2 , b_3 , and b_4 contain secular terms, i.e., the occurence of q_4^n $(n = 1, 2, \ldots)$. However, all of these terms occur in the product form: $\varepsilon^{n+1}\, q_4^n = \varepsilon(\varepsilon q_4)^n$. Therefore, since $|\varepsilon q_4| < 1$, these terms are at most of order ε , as are the other terms in the expressions.

Assuming $t_o = \theta_o = 0$, the classic results of Tsien, expressed in polar coordinates, are

$$
\begin{aligned}
r &= r_o[1 - \sqrt{\frac{r_o}{\mu}}\,(\frac{T}{m_o})\ t]^{-2} \\
\theta &= k(r/r_o - 1)/(r_o^2 T/m_o) \ .
\end{aligned}
\tag{4.24}
$$

When the b_4-equation and Equation (4.13) are evaluated at $t_o = \theta_o = 0$, with the small eccentricity assumption imposed, they match Equations (4.24). Thus, the Tsien solution is incorporated in the solution defined by $\{a(q, p), b(q, p)\}$, which also includes approximate Lagrange multiplier solutions.

The control angle, α , is defined by

$$\tan\alpha = \frac{B}{A} \ ,$$

where, as functions of q_i's and p_i's:

$$
\begin{aligned}
A &\equiv -q_1 p_1/\gamma + p_2[\cos q_4 + (q_2 + \cos q_4)/\gamma] + \\
&\quad + p_3[\sin q_4 + (q_3 + \sin q_4)/\gamma] \\
B &\equiv p_2 \sin q_4 - p_3 \cos q_4 \ .
\end{aligned}
$$

For true circumferential thrust, p_2 and p_3 would have to be identically zero, and since:

$$p_2 = a_2 = \text{constant}; \quad p_3 = a_3 = \text{constant},$$

then $a_2 = a_3 = 0$ would define this case. However, as previously stated, the reason for obtaining the approximate multiplier solutions is because of their differential equation properties and not their physical significance. That is, since the functional form of each of the Euler-Lagrange equations is

$$\dot{\lambda} = g(x, \lambda)$$

(i.e., they depend only on the control through the state), if the optimal state and circumferential thrust state are approximately the same, then the functional form of the solutions to the circumferential thrust multiplier equations should closely approximate the solutions of the Euler-Lagrange equations. However, both $p_2 = a_2$ and $p_3 = a_3$ are constant, which does not allow any variation in the multipliers as the state changes. This is an undesirable property of the solution which results from the fact that the defining Hamiltonian is of zero-order in q and s. Thus, since q and s are cyclic variables, their conjugate multipliers are constants of the motion. In the next section, this undesirable condition is improved.

4.2 A New Circumferential Thrust Approximation

In the previous section, a solution was obtained for the Hamiltonian system defined by:

$$K = [\Lambda_4 + h^3 k\Lambda_5/\gamma^2 + (T/m_o)(h^4k/\gamma^2)\sqrt{A^2 + B^2}\,]\Big|_{q=s=o}$$

As previously noted, this approximation is somewhat inadequate because K is cyclic in q and s. In Chapter 3, a closed-form Hamilton-Jacobi solution was obtained for the Hamiltonian

$$K_o = \Lambda_4 + h^3 k\Lambda_5/\gamma^2 \tag{4.25}$$

with no simplifying assumption for q and s. In this section, the solution for the Hamiltonian of Equation (4.25) will be perturbed into the solution for

$$K = \Lambda_4 + h^3 k\Lambda_5/\gamma^2 + [(T/m_o)(h^4k/\gamma^2)\sqrt{A^2 + B^2}]q = s = 0\ . \tag{4.26}$$

Since γ depends upon q and s, the conjugate multipliers of q and s will not be constants of the motion for this case.

In terms of the canonic constants for the coast-arc problem, Equations (3.74), the Hamiltonian of Equation (4.26) is

$$\begin{aligned} K = \alpha_1 &+ (kT/m_o)\alpha_2^4\ \{\alpha_2[-\beta_2 - 3\alpha_2^2\alpha_5 k\beta_1] \\ &+ 2[-\beta_3 + 2\alpha_2^3\alpha_5 k \sin\beta_1]\cos\beta_1 + \\ &+ 2[-\beta_4 - 2\alpha_2^3\alpha_5 k(1 + \cos\beta_1)]\} \end{aligned}$$

or

$$\begin{aligned} K = \alpha_1 &+ (kT/m_o)\alpha_2^4\ \{-\alpha_2\beta_2 - 2\beta_3\cos\beta_1 \\ &- 2\beta_4\sin\beta_1 - \alpha_2^3\alpha_5 k(3\beta_1 + 4\sin\beta_1)\}\ . \end{aligned} \tag{4.26'}$$

Since α_3 and α_4 are cyclic variables, it is desirable to have their conjugate variables as momenta in the H-J equation. Thus, the following simple canonical transformation is defined:

$$\begin{aligned} P_1 &= \alpha_1 , & P_2 &= \alpha_2 , & P_3 &= \beta_3 , & P_4 &= \beta_4 , & P_5 &= \alpha_5 \\ Q_1 &= \beta_1 , & Q_2 &= \beta_2 , & Q_3 &= -\alpha_3 , & Q_4 &= -\alpha_4 , & Q_5 &= \beta_5 . \end{aligned} \tag{4.27}$$

Then, the H-J equation for the Hamiltonian of Equation (4.26)', with θ as the independent variable, is

$$\begin{aligned} &\frac{\partial S}{\partial \theta} + \frac{\partial S}{\partial Q_1} + (kT/m_o) \left(\frac{\partial S}{\partial Q_2}\right)^4 \left[-Q_2 \frac{\partial S}{\partial Q_2} - \right. \\ &\left. - 2 \frac{\partial S}{\partial Q_3} \cos Q_1 - 2 \frac{\partial S}{\partial Q_4} \sin Q_1 - k\left(\frac{\partial S}{\partial Q_2}\right)^3 \frac{\partial S}{\partial Q_5} (3Q_1 + 4\sin Q_1)\right] = 0 . \end{aligned} \tag{4.28}$$

Since neither θ , Q_3 , Q_4 , nor Q_5 appear explicitly in Equation (4.28), a partial separation of variables can be effected. Thus, assume

$$S = S_o(\theta) + S_1(q_3) + S_2(Q_4) + S_3(Q_5) + S^*(Q_1, Q_2) . \tag{4.29}$$

After substituting Equation (4.29) into Equation (4.28) and applying the usual independence arguments, the following four constants of the solution are defined:

$$A_1 = \frac{\partial S}{\partial \theta} , \quad A_2 = \frac{\partial S}{\partial Q_3} , \quad A_3 = \frac{\partial S}{\partial Q_4} , \quad A_4 = \frac{\partial S}{\partial Q_5} .$$

Thus,

$$S \;=\; A_1\theta + A_2Q_3 + A_3Q_4 + A_4Q_5 + S^*(Q_1,\, Q_2)\;. \tag{4.30}$$

Equation (4.28) can then be reduced to the following partial differential equation in two independent variables

$$\begin{aligned}\psi(Q_1,\, Q_2) \;\equiv\; & A_1 + \frac{\partial S^*}{\partial Q_1} + (kT/m_o)\left(\frac{\partial S^*}{\partial Q_2}\right)^4 \Big[-Q_2\,\frac{\partial S^*}{\partial Q_2} \\ & - 2A_2\cos Q_1 - 2A_3\sin Q_1 - kA_4\left(\frac{\partial S^*}{\partial Q_2}\right)^3 (3Q_1 + 4\sin Q_1)\Big] = 0\;.\end{aligned} \tag{4.31}$$

Inspection of the characteristic system for Equation (4.31) gives the remaining constant for the complete solution, i.e.,

$$\begin{aligned}&\frac{d\left(\frac{\partial S^*}{\partial Q_2}\right)}{d\tau} \qquad \frac{\partial\psi}{\partial Q_2} - (kT/m_o)\left(\frac{\partial S^*}{\partial Q_2}\right)^5 \\ &\frac{dQ_1}{d\tau} \;=\; 1\;.\end{aligned} \tag{4.32}$$

The form of these equations is the same as the form of Equations (4.11), so the following result is obtained by comparison with Equation (4.13)

$$\frac{\partial S^*}{\partial Q_2} \;=\; A_5(1 - \varepsilon Q_1)^{-\frac{1}{4}}\;, \tag{4.33}$$

where $\varepsilon \equiv 4A_5^4(kT/m_o)$. In comparison with the complete solution of Section (4.1), $A_4 = a_4$ and $A_5 = a_5$. The remaining constants are not equal.

Since S^* is a function of Q_1 and Q_2 , the following Pfaff differential equation must be integrated

$$dS^* = \frac{\partial S^*}{\partial Q_1}(Q_1, Q_2)dQ_1 + \frac{\partial S^*}{\partial Q_2}(Q_1)dQ_2 \ , \tag{4.34}$$

where $\frac{\partial S^*}{\partial Q_1}$ is obtained by substituting Equation (4.33) into Equation (4.31). As in Section (4.1), since $\frac{\partial S^*}{\partial Q_2}$ is a function of Q_1 alone, the solution of Equation (4.34) must be of the form

$$S^* = \frac{\partial S^*}{\partial Q_2} Q_2 + \int \left(\frac{\partial S^*}{\partial Q_1}\right)' dQ_1 \tag{4.35}$$

where $\left(\frac{\partial S^*}{\partial Q_1}\right)'$ is that portion of $\frac{\partial S^*}{\partial Q_1}$ which does not contain Q_2 , i.e.,

$$\int \left(\frac{\partial S^*}{\partial Q_1}\right)' dQ_1 = -A_1Q_1 + (\varepsilon/4)\int \{2A_2(1 - \varepsilon Q_1)^{-1}\cos Q_1$$
$$+ 2A_3(1 - \varepsilon Q_1)^{-1}\sin Q_1 + kA_4A_5^3(1 - \varepsilon Q_1)^{-7/4}(3Q_1 + 4\sin Q_1)\} dQ_1 \ . \tag{4.36}$$

Comparison of Equation (4.36) with Equation (4.18) shows that the only new terms to be integrated are

$$\varepsilon^2A_4/(15A_5T/m_o)\ \{3\int Q_1(1 - \varepsilon Q_1)^{-7/4}dQ_1 + 4\int (1 - \varepsilon Q_1)^{-7/4} \sin Q_1\ dQ_1\}. \tag{4.37}$$

The first integral can be evaluated in closed-form

$$\int Q_1(1 - \varepsilon Q_1)^{-7/4}\ dQ_1 = 4(4 - 3\varepsilon Q_1)\ (1 - \varepsilon Q_1)^{-3/4}/3\varepsilon^2 \ . \tag{4.38}$$

The second integral cannot be evaluated in closed-form but $|\varepsilon Q_1| < 1$ so a Taylor series for $(1 - \varepsilon Q_1)^{-7/4}$ can be used to form a series of terms which are integrable. However, to obtain a more rapidly convergent series, it is beneficial to first perform two integrations by parts which give:

$$\int (1 - \varepsilon Q_1)^{-7/4} \sin Q_1 \, dQ_1 = (16/3\varepsilon^2) \int (1 - \varepsilon Q_1)^{1/4} \sin Q_1 \, dQ_1$$
$$+ [4\varepsilon(\sin Q_1 - 4Q_1 \cos Q_1) + 16\cos Q_1] (1 - \varepsilon Q_1)^{-3/4}/(3\varepsilon^2) \ . \tag{4.39}$$

The Taylor series for $(1 - \varepsilon Q_1)^{1/4}$, expanded about $\varepsilon Q_1 = C^*$ with $L \equiv (1 - C^*)^{1/4}$, is

$$(1 - \varepsilon Q_1)^{1/4} = L - (\varepsilon Q_1 - C^*)/(4L^3) -$$
$$- \sum_{n=2}^{\infty} \frac{3 \cdot 7 \ldots [3 + 4(n - 2)]}{n! 4^n \, L^{[3 + 4(n - 1)]}} (\varepsilon Q_1 - C^*)^n \ . \tag{4.40}$$

Then after making use of the results of Section (4.1) and integrating Equation (4.39), Equation (4.36) becomes

$$\int \left(\frac{\partial S^*}{\partial Q_1} \right)' dQ_1 = -A_1 Q_1 +$$
$$+ A_4[4(3 + 4\cos Q_1) + \varepsilon(4\sin Q_1 - 16Q_1 \cos Q_1 - 9Q_1)] (1 - \varepsilon Q_1)^{-3/4} \cdot$$
$$\cdot (12A_5T/m_o)^{-1} + \frac{A_2}{2} \{\varepsilon N_o \sin Q_1 +$$
$$+ \varepsilon^2 N_1(\cos Q_1 + Q_1 \sin Q_1) + \varepsilon^3 N_2[2Q_1 \cos Q_1$$
$$+ (Q_1^2 - 2)\sin Q_1] + \ldots\} + \frac{A_3}{2} \{-\varepsilon \ N_o \cos Q_1 \tag{4.41}$$
$$+ \varepsilon^2 \ N_1(\sin Q_1 - Q_1 \cos Q_1) + \varepsilon^3 N_2[2Q_1 \sin Q_1$$
$$+ (2 - Q_1^2)\cos Q_1] + \ldots\} + (4A_4/(3A_5T/m_o)) \cdot$$
$$\cdot \{-M_o \cos Q_1 + \varepsilon M_1(\sin Q_1 - Q_1 \cos Q_1) +$$
$$+ \varepsilon^2 M_2[2Q_1 \sin Q_1 + (2 - Q_1^2)\cos Q_1] +$$
$$+ \varepsilon^3 M_3[(3Q_1^2 - 6)\sin Q_1 + (6Q_1 - Q_1^3)\cos Q_1] + \ldots\} ,$$

where the N_i's are defined by Equations (4.21) and

$$M_o \equiv L + \frac{C^*}{4L^3}\left[1 - \sum_{n=2}^{\infty} \frac{3\cdot 7\ldots[3+4(n-2)]}{n!L^{4(n-1)}}\left(\frac{-C^*}{4}\right)^{n-1}\right]$$

$$M_1 \equiv -\frac{1}{4L^3}\left[1 + \sum_{n=2}^{\infty} \frac{3\cdot 7\ldots[3+4(n-2)]}{(n-1)!L^{4(n-1)}}\left(\frac{-C^*}{4}\right)^{n-1}\right] \tag{4.42}$$

$$M_2 \equiv \frac{-3}{2!4^2L^7} + \frac{3(3\cdot 7)}{3!4^3L^{11}}C^* - \frac{6(3\cdot 7\cdot 11)}{4!4^4L^{15}}(C^*)^2 + - \ldots$$

$$M_3 \equiv -\frac{3\cdot 7}{3!4^3L^{11}} + \frac{5(3\cdot 7\cdot 11)}{4!4^4L^{15}}C^* - \frac{15(3\cdot 7\cdot 11\cdot 15)}{5!4^5L^{19}}(C^*)^2 + - \ldots .$$

The M_i-expressions are determined by the Taylor series expansion of $(1 - \varepsilon Q_1)^{\frac{1}{4}}$, with M_i representing the coefficient of $(\varepsilon Q_1)^i$.

Equations (4.32), (4.35), and (4.41) can be combined to give the solution of Equation (4.28)

$$\begin{aligned} S = {} & A_1\theta + A_2Q_3 + A_3Q_4 + A_4Q_5 + \\ & + A_5Q_2(1 - \varepsilon Q_1)^{-\frac{1}{4}} + \int \left(\frac{\partial S^*}{\partial Q_1}\right)' dQ_1 . \end{aligned} \tag{4.43}$$

The remaining canonic constants are formed by applying Jacobi's theorem and making use of the fact that $\frac{\partial \varepsilon}{\partial A_5} = 4\varepsilon/A_5$:

$$\begin{aligned} B_1 &= \theta - Q_1 \\ B_2 &= Q_3 + \frac{\varepsilon}{2}\{N_o \sin Q_1 + \varepsilon N_1(\cos Q_1 + Q_1 \sin Q_1) + \ldots\} \\ B_3 &= Q_4 + \frac{\varepsilon}{2}\{-N_o \cos Q_1 + \varepsilon N_1(\sin Q_1 - Q_1 \cos Q_1) + \ldots\} \\ B_4 &= Q_5 + (1 - \varepsilon Q_1)^{-3/4}\,[4(3 + 4\cos Q_1) + \varepsilon(4\sin Q_1 \\ &\quad - 16Q_1\cos Q_1 - 9Q_1)]/(12A_5T/m_o) + \\ &\quad + 4\{-M_o\cos Q_1 + \varepsilon M_1(\ldots) + \varepsilon^2 M_2[\ldots] + \ldots\}/(3A_5T/m_o) \end{aligned} \tag{4.44}$$

$$B_5 = Q_2(1 - \varepsilon Q_1)^{-5/4} + A_4(1 - \varepsilon Q_1)^{-7/4} \cdot$$

$$\cdot \; [3\varepsilon(3Q_1 + 4\sin Q_1) - 4(1 - \varepsilon Q_1)(3 + 4\cos Q_1)]/(12A_5^2 T/m_o)$$

$$+ \; \frac{2A_2}{A_5}\{\varepsilon N_o \sin Q_1 + 2\varepsilon^2 N_1(\ldots) + \ldots\}$$

$$+ \; \frac{2A_3}{A_5}\{-\varepsilon N_o \cos Q_1 + 2\varepsilon^2 N_1(\ldots) + \ldots\}$$

$$- \; [4A_4/(3A_5^2 T/m_o)]\{-M_o \cos Q_1 + \varepsilon M_1(\ldots) + \ldots\}$$

$$+ \; [16A_4/(3A_5^2 T/m_o)]\{\varepsilon M_1(\ldots) + 2\varepsilon^2 M_2[\ldots] + \ldots\} \; ,$$

where $(\ldots)$ and $[\ldots]$ represent the bracketed terms in Equation (4.41) which are multiplied by A_2N_i , A_3N_i , and M_i $(i = 1, 2, \ldots)$.

As previously noted, the Λ_2 and Λ_3 solutions (defined by the Λ_2 and Λ_3 equations) are not constant functions in this system whereas they are in the system of Section (4.1). Another important difference in the two solutions is the $t(\theta)$ - relationship defined by the b_4 - and B_4 - equations. The difference is seen more clearly if B_4 is developed before evaluating the second integral of Equation (4.37), i.e.,

$$B_4 = Q_5 + (4 - 3\varepsilon Q_1)(1 - \varepsilon Q_1)^{-3/4}/(4A_5 T/m_o) + \varepsilon k A_5^3 \int (1 - \varepsilon Q_1)^{-7/4} \sin Q_1 \, dQ_1 \; . \tag{4.45}$$

Also, recall from Equations (4.23) and Equation (3.72)

$$b_4 = q_5 + (1 - \varepsilon q_4)^{1/4}/(a_5 T/m_o) \tag{4.46}$$

$$Q_5 \equiv \beta_5 = q_5 - \alpha_2^3 k q_4 + O(q, s) \; . \tag{4.47}$$

After substituting Equation (4.47) into Equation (4.45) and rewriting Equations (4.45) and (4.46) in terms of the original Poincare variables, the following results are obtained

$$B_4 = t - kh^3\theta + (4 - 3\varepsilon\theta)(1 - \varepsilon\theta)^{-3/4}/(4h_o T/m_o)$$
$$+ \varepsilon k h_o^3 \int (1 - \varepsilon Q_1)^{-7/4} \sin Q_1 dQ_1$$

and,

$$b_4 = t + (1 - \varepsilon\theta)^{1/4}/(h_o T/m_o) \ . \quad (4.48)$$

But, from Equation (4.33)

$$h^3 = h_o^3(1 - \varepsilon\theta)^{-3/4} \ ,$$

so

$$B_4 = t - kh_o^3\theta(1 - \varepsilon\theta)^{-3/4} + (4 - 3\varepsilon\theta)(1 - \varepsilon\theta)^{-3/4}/(4h_o T/m_o)$$
$$+ \varepsilon k h_o^3 \int (1 - \varepsilon Q_1)^{-7/4} \sin Q_1 dQ_1 \ .$$

After combining the $(1 - \varepsilon\theta)^{-3/4}$ terms

$$B_4 = t + (1 - \varepsilon\theta)^{1/4}/(h_o T/m_o) + O(\varepsilon) \ . \quad (4.49)$$

Thus, to order ε, the $t(\theta)$-relationships of the two sections are also identical (i.e., Equations (4.48) and (4.49)). For small θ, Equations (4.48) and (4.49) are nearly the same. Since the indefinite integral of the B_4-expression cannot be evaluated in closed-form, the advantage of using B_4 in of b_4 depends on the validity of the approximation for this integral. A numerical comparison of the two expressions is given in Chapter 7.

CHAPTER 5

NON-HAMILTONIAN CIRCUMFERENTIAL THRUST ANALYSIS

In this Chapter, asymptotic solutions of the circumferential thrust problem which are more general than the solutions of Chapter 4 will be presented. The governing differential equations will be developed in Poincare variables and then certain simplifying assumptions will be made. Due to the large number of various approximate circumferential thrust state solutions[32], no claim of originiality is made with respect to the solutions for the state variables. However, it appears that the associated Lagrange multiplier solutions are new.

In the following sections, the state equations will be solved with the small eccentricity assumption. However, the mass will be represented by a binomial series, and second order solutions in $\delta = \frac{\dot{m}_o}{m_o}$ for $h(\theta; \delta)$ and $t(\theta; \delta)$ will be formed. These solutions are very useful in predicting the time duration of optimal multirevolution circular orbit transfers. The method of solution for $q(\theta; \varepsilon, \delta)$ and $s(\theta; \varepsilon, \delta)$ will be indicated. These solutions can be represented as series in two parameters, δ and $\varepsilon \equiv 4h_o^4 \frac{T}{m_o} k$.

The Lagrange multiplier solutions will be solved with both the small eccentricity and the constant mass assumptions. The solutions are developed to first-order in ε , however all orders of solution can be obtained by simple quadratures.

The analysis of this chapter was developed to support the numerical work of Chapter 7, and also to serve as a guide for future Hamiltonian studies involving the circumferential thrust problem. Thus, convergence properties and qualitative aspects of the solutions are not fully discussed. However, in Chapter 7, the solutions are compared to the true solution for a representative mission.

5.1 The Solution of the State Equations

Consider the generalized Hamiltonian in Poincare variables with time as the independent variable, and without treating time as a coordinate, i.e.,

$$K = \Lambda_4 \, \gamma^2/(h^3 k) + \frac{T}{m} h \sqrt{A^2 + B^2} \, .$$

The circumferential thrust Hamiltonian is defined by requiring $B \equiv 0$, so

$$K = \Lambda_4 \, \gamma^2/(h^3 k) + \frac{T}{m} hA \, . \tag{5.1}$$

The differnetial equations for the state variables, with the small eccentricity assumption, are of the form $\frac{dX_i}{dt} = \left.\frac{\partial K}{\partial \Lambda_i}\right|_{q=s=o}$. Since the right-hand sides of these equations are defined by derivatives of the Hamiltonian with respect to multipliers, then the small eccentricity assumption can be made in the Hamiltonian before differentiation, i.e., $\frac{\partial K\big|_{q=s=o}}{\partial \Lambda_i} \equiv \left.\frac{\partial K}{\partial \Lambda_i}\right|_{q=s=o}$. Thus,

$$K \simeq \Lambda_4/(h^3 k) + \frac{T}{m} h[\Lambda_1 h + 2\Lambda_2 \cos\theta + 2\Lambda_3 \sin\theta] \tag{5.2}$$

The Hamilton's equations for the state variables are

$$\frac{dh}{dt} = \frac{\partial K}{\partial \Lambda_1} = \frac{T}{m} h^2$$

$$\frac{dq}{dt} = \frac{\partial K}{\partial \Lambda_2} = 2\frac{T}{m} h \cos\theta$$

$$\frac{ds}{dt} = \frac{\partial K}{\partial \Lambda_3} = 2\frac{T}{m} h \sin\theta$$

$$\frac{d\theta}{dt} = \frac{\partial K}{\partial \Lambda_4} = 1/(h^3 k) \, .$$

Then, since $\frac{d\theta}{dt} = 1/(h^3k)$, the rates of change of h , q , s , and t with respect to θ are given by

$$\begin{aligned}
\frac{dh}{d\theta} &= \frac{T}{m} k\, h^5 \\
\frac{dq}{d\theta} &= 2\,\frac{T}{m} k\, h^4 \cos\theta \\
\frac{ds}{d\theta} &= 2\,\frac{T}{m} k\, h^4 \sin\theta \\
\frac{dt}{d\theta} &= h^3k\ .
\end{aligned} \tag{5.3}$$

Let $\delta \equiv \frac{\dot{m}_o}{m_o}$. Then, assuming $t_o = 0$,

$$\frac{1}{m} = \frac{1}{m_o}(1 + \delta t)^{-1}$$

or

$$\frac{1}{m} = \frac{1}{m_o}(1 - \delta t + \delta^2 t^2 - + \ldots)\ . \qquad (|\delta t| < 1)$$

The differential equations for h and t , to second-order in δ , are then

$$\begin{aligned}
\frac{dh}{d\theta} &= \frac{T}{m_o}(1 - \delta t + \delta^2 t^2)k\, h^5 \\
\frac{dt}{d\theta} &= h^3k\ .
\end{aligned} \tag{5.4}$$

Assume solutions of the following form

$$h = h^o + \delta h^1 + \delta^2 h^2 + \dots$$
$$t = t^o + \delta t^1 + \delta^2 t^2 + \dots \quad . \tag{5.5}$$

After substituting Equations (5.5) into Equations (5.4) and equating coefficients of like powers of δ , the following perturbation equations are formed

$$\frac{dh^o}{d\theta} = \frac{T}{m_o} k(h^o)^5$$
$$\frac{dt^o}{d\theta} = (h^o)^3 k \ ; \tag{5.6}$$

$$\frac{dh^1}{d\theta} = \frac{T}{m_o} k[5(h^o)^4 \ h^1 - (h^o)^5 \ t^o]$$
$$\frac{dt^1}{d\theta} = 3k(h^o)^2 \ h^1 \ ; \tag{5.7}$$

and

$$\frac{dh^2}{d\theta} = \frac{T}{m_o} k[5(h^o)^4 \ h^2 + 10(h^o)^3 \ (h^1)^2 - (h^o)^5 \ t^1$$
$$- \ 5(h^o)^4 \ t^o \ h^1 + (h^o)^5 \ (t^o)^2] \tag{5.8}$$
$$\frac{dt^2}{d\theta} = 3k[(h^o)^2 \ h^2 + 3kh^o(h^1)^2] \ .$$

Note that the perturbation equations must be solved in the order that they are written.

The first equation of Equation (5.6) can be solved by separation of variables, and its solution can be expressed as

$$h^o = h_o(1 - \varepsilon\theta)^{-\frac{1}{4}} , \tag{5.9}$$

where

$$\theta(t_o) = 0 , \quad h_o \equiv h(t_o) , \quad \varepsilon \equiv 4h_o^4 \frac{T}{m_o} k . \tag{5.10}$$

Substitution of Equation (5.9) into the second of Equations (5.6) defines another equation solvable by separation of variables with the result

$$t^o = [1 - (1 - \varepsilon\theta)^{\frac{1}{4}}]/(h_o \frac{T}{m_o}) . \tag{5.11}$$

Equations (5.9) and (5.11) represent the zero-order solution and they are the same as the solution by Tsien[44] discussed in Chapter 4.

To obtain the first-order solutions, Equations (5.9) and (5.11) must be substituted into Equations (5.7). Then, the first of Equations (5.7) can be written as

$$\frac{dh^1}{d\theta} - \frac{5}{4}\varepsilon(1 - \varepsilon\theta)^{-1} h^1 = kh_o^4[(1 - \varepsilon\theta)^{-1} - (1 - \varepsilon\theta)^{-5/4}] . \tag{5.12}$$

Equation (5.12) is a linear, inhomogeneous differential equation whose homogeneous solution is

$$(h^1) = C_o(1 - \varepsilon\theta)^{-5/4} ,$$

where C_o is a constant of integration. A particular solution of Equation (5.12) can be obtained by assuming the following solution form:

$$h^1 = C_1 + C_2(1 - \varepsilon\theta)^{-1/4} .$$

Then, after applying the boundary condition $h^1(0) = 0$, the solution for h^1 is easily formed, i.e.,

$$h^1 = [5(1 - \varepsilon\theta)^{-1/4} - (1 - \varepsilon\theta)^{-5/4} - 4]/(20\ T/m_o) . \tag{5.13}$$

The differential equation for t^1 , upon making use of Equation (5.13), is

$$\frac{dt^1}{d\theta} = 3kh_o^2[5(1 - \varepsilon\theta)^{-3/4} - (1 - \varepsilon\theta)^{-7/4} - 4(1 - \varepsilon\theta)^{-1/2}]/(20\ T/m_o) .$$

This equation integrates to give

$$t^1 = [10 - 15(1 - \varepsilon\theta)^{1/4} + 6(1 - \varepsilon\theta)^{1/2} - (1 - \varepsilon\theta)^{-3/4}]/[20(h_o T/m_o)^2] . \tag{5.14}$$

Thus, the solution of Equations (5.3) to first-order in δ is given by

$$\begin{aligned} h &= h^o + \delta h^1 \\ t &= t^o + \delta t^1 , \end{aligned} \tag{5.15}$$

where Equations (5.9), (5.11), (5.13), and (5.14) define the right-hand sides of Equations (5.15).

After substituting the first and second-order solutions into the first of Equations (5.8) and combining terms, the following differential equation is formed

$$\frac{dh^2}{d\theta} - \frac{5}{4}\,\varepsilon(1-\varepsilon\theta)^{-1}h^2 = kh_o^3[(1-\varepsilon\theta)^{-13/4} - 5(1-\varepsilon\theta)^{-5/4} + 4(1-\varepsilon\theta)^{-3/4}]/(40\ T/m_o)\ . \tag{5.16}$$

Again this is a linear inhomogeneous equation whose homogeneous solution is of the form

$$(h^2) = C_o(1-\varepsilon\theta)^{-5/4}\ .$$

A particular solution can be obtained by assuming the following solution form:

$$h^2 = C_1(1-\varepsilon\theta)^{-9/4} + C_2(1-\varepsilon\theta)^{-1/4} + C_3(1-\varepsilon\theta)^{1/4}\ .$$

After the applying the boundary condition, the solution of Equation (5.16) is given by

$$h^2 = [(1-\varepsilon\theta)^{-9/4}/160 - (1-\varepsilon\theta)^{-5/4}/48 + (1-\varepsilon\theta)^{-1/4}/32 - (1-\varepsilon\theta)^{1/4}/60]/(h_o T^2/m_o^2)\ . \tag{5.17}$$

The t^2 - differential equation is then

$$\frac{dt^2}{d\theta} = [3k\ h_o/(T/m_o)^2]\ [7(1-\varepsilon\theta)^{-11/4}/800 - 11(1-\varepsilon\theta)^{-7/4}/240 + (1-\varepsilon\theta)^{-3/2}/50 + 3(1-\varepsilon\theta)^{-3/4}/32 - (1-\varepsilon\theta)^{-1/2}/10 + 7(1-\varepsilon\theta)^{-1/4}/300]\ .$$

This equation integrates to give

$$
\begin{aligned}
t^2 \;=\; & \tfrac{3}{4}[2/9 + (1 - \varepsilon\theta)^{-7/4}/200 - 11(1 - \varepsilon\theta)^{-3/4}/180 \\
& + (1 - \varepsilon\theta)^{-\frac{1}{2}}/25 - 3(1 - \varepsilon\theta)^{\frac{1}{4}}/8 + (1 - \varepsilon\theta)^{\frac{1}{2}}/5 \\
& - 7(1 - \varepsilon\theta)^{3/4}/225]/(h_o T/m_o)^3 \;.
\end{aligned}
\qquad (5.18)
$$

Therefore, to second-order in δ , the solution of Equations (5.3) is

$$
\begin{aligned}
h &= h^o + \delta h^1 + \delta^2 h^2 \\
t &= t^o + \delta t^1 + \delta^2 t^2 \;.
\end{aligned}
$$

These equations are very useful for determining approximate cutoff-times in numerical iteration schemes involving circular orbit transfers. For example, given the initial and final values of the radii of the two circular orbits, initial and final values of h can be determined. Then, Equation (5.9) can be used to solve for a zero-order approximation of θ_f in closed form. Upon substitution of this value in Equation (5.11), a zero-order approximation of the final time can be determined, say t_f^o . Then, a simple Newton's iteration scheme, which uses t_f^o as a first guess, can be used to determine the second-order approximation of the final time, say t_f^2 . For all of the numerical studies which were performed in support of this thesis, the difference between the optimal final time and t_f^2 was always less than 1.0% (see Table 2).

The differential equations for q and s will now be considered. Equations (5.3)

$$\frac{dq}{d\theta} = 2\,\frac{T}{m}\,kh^4 \cos\theta$$

$$\frac{ds}{d\theta} = 2\,\frac{T}{m}\,kh^4 \cos\theta$$

or,

$$\begin{aligned}\frac{dq}{d\theta} &= 2\,\frac{T}{m_o}\,k(1 - \delta t + \delta^2 t^2)\,(h^o + \delta h^1 + \delta h^2)\cos\theta \\ \frac{ds}{d\theta} &= 2\,\frac{T}{m_o}\,k(1 - \delta t + \delta^2 t^2)\,(h^o + \delta h^1 + \delta h^2)\sin\theta\ ,\end{aligned} \qquad (5.19)$$

to second-order in δ .

Assume solutions of the form

$$\begin{aligned} q &= q^o + \delta q^1 + \delta^2 q^2 \\ s &= s^o + \delta s^1 + \delta^2 s^2\ . \end{aligned} \qquad (5.20)$$

After substituting Equations (5.20) into Equations (5.19), the following perturbation equations are formed:

$$\frac{dq^o}{d\theta} = 2k\,\frac{T}{m_o}\,(h^o)^4 \cos\theta$$

$$\frac{dq^1}{d\theta} = 2k\,\frac{T}{m_o}\,[4\,(h^o)^3 h^1 - (h^o)^4 t^o]\cos\theta$$

$$\begin{aligned}\frac{dq^2}{d\theta} &= 2k\,\frac{T}{m_o}\,\{(h^o)^4[(t^o)^2 - t^1] + 4(h^o)^3[h^2 - h^1 t^o] \\ &\quad + 6(h^o)^2(h^1)^2\}\cos\theta\ ,\end{aligned} \qquad (5.21)$$

and similarly for $\frac{ds^i}{d\theta}$ $(i = 0, 1, 2)$ with sine θ replacing cosine θ.

Equations (5.21) are easily integrated if one assumes Taylor series expansions for the $(1 - \varepsilon\theta)^m$-terms which appear throughout Equations (5.21). The resultant solutions will then be power series in both ε and δ. However, the series should be rapidly convergent since all terms containing θ^n are multiplied by ε^{n+1}.

5.2 The Solution of the Lagrange Multiplier Equations

In Chapter 4 it was argued that even though the Lagrange multipliers do not have physical significance when circumferential thrust is specified, they may be approximated by the circumferential multiplier solutions if the optimal state variables are closely approximated by the circumferential state variables. Thus, approximate circumferential multiplier solutions may be applicable in open-loop guidance analysis.

Consider the Hamiltonian of Equation (5.1). Since the differential equations for the multipliers are defined by $\frac{d\Lambda_i}{dt} = -\frac{\partial K}{\partial X_i}$ $(i = 1,\ldots,4)$, the small eccentricity assumption cannot be made in the Hamiltonian. That is, the differential equations must be formed first and then the $q = s = 0$ assumption can be applied. Otherwise, $\frac{\partial K}{\partial q} = \frac{\partial K}{\partial s} = 0$ which implies Λ_2 and Λ_3 are constant.

After substituting for A and assuming $m = m_o$, Equation (5.1) can be written as

$$K = \Lambda_4\, \gamma^2/(h^3 k) + \frac{T}{m_o}\, h\{\Lambda_1 h/\gamma + \Lambda_2[\cos\theta + (q + \cos\theta)/\gamma] + \Lambda_3[\sin\theta + (s + \sin\theta)/\gamma]\}. \tag{5.22}$$

Then, the Hamilton's equations for the multipliers are

$$\frac{d\Lambda_1}{dt} = -\frac{\partial K}{\partial h} = (-2\frac{T}{m_o} h/\gamma)\Lambda_1 - \frac{T}{m_o}[\cos\theta + (q + \cos\theta)/\gamma]\Lambda_2$$
$$- \frac{T}{m_o}[\sin\theta + (s + \sin\theta)/\gamma]\Lambda_3 + [3\gamma^2/(h^4 k)]\Lambda_4$$

$$\frac{d\Lambda_2}{dt} = -\frac{\partial K}{\partial q} = (\frac{T}{m_o} h/\gamma)\ \{\Lambda_1 h \cos\theta/\gamma - \Lambda_2[1$$
$$- (q + \cos\theta)\cos\theta/\gamma] + \Lambda_3(s + \sin\theta)\cos\theta/\gamma\}$$
$$- [2\gamma \cos\theta/(h^3 k)]\Lambda_4$$

(5.23)

$$\frac{d\Lambda_3}{dt} = -\frac{\partial K}{\partial s} = (\frac{T}{m_o} h/\gamma)\ \{\Lambda_1 h \sin\theta/\gamma +$$
$$+ \Lambda_2(q + \cos\theta)\sin\theta/\gamma - \Lambda_3[1 - (s + \sin\theta)\sin\theta/\gamma]\}$$
$$- [2\gamma \sin\theta/(h^3 k)]\Lambda_4$$

$$\frac{d\Lambda_4}{dt} = -\frac{\partial K}{\partial \theta} = \frac{T}{m_o} h[(1 + 1/\gamma)\sin\theta\ \Lambda_2 - (1 + 1/\gamma)\cos\theta\ \Lambda_3]$$
$$+ (s\cos\theta - q\sin\theta)\ [-2\Lambda_4\gamma\ /(h^3 k) + \frac{T}{m_o}\Lambda_1 h^2/\gamma^2$$
$$+ \frac{T}{m_o}\Lambda_2 h(q + \cos\theta)/\gamma^2 + \frac{T}{m_o}\Lambda_3 h(s + \sin\theta)/\gamma^2]\ .$$

After changing the independent variable from t to θ and making the small eccentricity assumption, Equations (5.23) become

$$\frac{d\Lambda_1}{d\theta} = (-2\frac{T}{m_o} h^4 k)\Lambda_1 - (2\frac{T}{m_o} h^3 k \cos\theta)\Lambda_2 -$$
$$- (2\frac{T}{m_o} h^3 k \sin\theta)\Lambda_3 + (3/h)\Lambda_4$$

$$\frac{d\Lambda_2}{d\theta} = (\frac{T}{m_o} h^4 k) [(h \cos\theta)\Lambda_1 - (\sin^2\theta)\Lambda_2 + (\sin\theta \cos\theta)\Lambda_3] - (2\cos\theta)\Lambda_4$$

$$\frac{d\Lambda_3}{d\theta} = (\frac{T}{m_o} h^4 k) [(h \sin\theta)\Lambda_1 + (\sin\theta \cos\theta)\Lambda_2 - (\cos^2\theta)\Lambda_3] - (2\sin\theta)\Lambda_4 \tag{5.24}$$

$$\frac{d\Lambda_4}{d\theta} = \frac{T}{m_o} h^4 k[(2\sin\theta)\Lambda_2 - (2\cos\theta)\Lambda_3] .$$

The system of Equations (5.24) is a linear system with variable coefficients, where $h(\theta)$ is given by Equation (5.9) (i.e., $h \equiv h^o$) since $f = 0$. Thus, in terms of ε, Equations (5.24) become

$$\frac{d\Lambda_1}{d\theta} = -[\varepsilon/(4h_o)] (1 - \varepsilon\theta)^{-3/4} \{2h_o(1 - \varepsilon\theta)^{-1/4}\Lambda_1 + 2\cos\theta\, \Lambda_2 + 2\sin\theta\, \Lambda_3\} + 3(1 - \varepsilon\theta)^{1/4}\Lambda_4/h_o$$

$$\frac{d\Lambda_2}{d\theta} = (\varepsilon/4) (1 - \varepsilon\theta)^{-1} \{h_o \cos\theta(1 - \varepsilon\theta)^{-1/4}\Lambda_1 - \sin^2\theta\, \Lambda_2 + \cos\theta \sin\theta\, \Lambda_3\} - 2\cos\theta\, \Lambda_4 \tag{5.25}$$

$$\frac{d\Lambda_3}{d\theta} = (\varepsilon/4) (1 - \varepsilon\theta)^{-1} \{h_o \sin\theta(1 - \varepsilon\theta)^{-1/4}\Lambda_1 + \sin\theta \cos\theta\, \Lambda_2 - \cos^2\theta\, \Lambda_3\} - 2\sin\theta\, \Lambda_4$$

$$\frac{d\Lambda_4}{d\theta} = (\varepsilon/4) (1 - \varepsilon\theta)^{-1} \{2\sin\theta\, \Lambda_2 - 2\cos\theta\, \Lambda_3\} .$$

Since Equations (5.25) are easily integrated when $\varepsilon = 0$, assume solutions of the form

$$\Lambda_i = \Lambda_i^o + \varepsilon\Lambda_i^1 + \varepsilon^2\Lambda_i^2 + \ldots \quad (i = 1,\ldots,4) \tag{5.26}$$

Further, since only missions where $|\varepsilon\theta| < 1$ can be considered, assume binomial expansions for all of the $(1 - \varepsilon\theta)^n$ - terms which appear in Equations (5.24). Then, the resultant zero and first-order perturbation equations are.

$$\begin{aligned}
\frac{d\Lambda_1^o}{d\theta} &= 3\Lambda_4^o/h_o \\
\frac{d\Lambda_2^o}{d\theta} &= (-2\cos\theta)\Lambda_4^o \\
\frac{d\Lambda_3^o}{d\theta} &= (-2\sin\theta)\Lambda_4^o \\
\frac{d\Lambda_4^o}{d\theta} &= 0\ ;
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
\frac{d\Lambda_1^1}{d\theta} &= (3/h_o)\Lambda_4^1 - [2h_o\Lambda_1^o + 2\cos\theta\ \Lambda_2^o \\
&\quad + 2\sin\theta\ \Lambda_3^o + 3\theta\Lambda_4^o]/4h_o \\
\frac{d\Lambda_2^1}{d\theta} &= (-2\cos\theta)\Lambda_4^1 + [h_o\cos\theta\ \Lambda_1^o - \sin^2\theta\ \Lambda_2^o \\
&\quad + \cos\theta\sin\theta\ \Lambda_3^o]/4 \\
\frac{d\Lambda_3^1}{d\theta} &= (-2\sin\theta)\Lambda_4^1 + [h_o\sin\theta\ \Lambda_1^o + \cos\theta\sin\theta\ \Lambda_2^o \\
&\quad - \cos^2\theta\ \Lambda_3^o]/4
\end{aligned} \tag{5.28}$$

$$\frac{d\Lambda_4^1}{d\theta} = (\sin\theta\ \Lambda_2^o - \cos\theta\ \Lambda_3^o)/2 .$$

Note that the only first-order term which appears on the right-hand sides of Equations (5.28) is Λ_4^1 ; and the differential equation for Λ_4^1 contains only zero-order terms on its right-hand side. Because of this property, all orders of solution can be obtained by simple quadratures. That is, the n^{th}-order perturbation equations for Λ_1^n , Λ_2^n , and Λ_3^n will depend upon Λ_4^n , but the differential equation for Λ_4^n will depend at most upon the Λ_i^{n-1} (i = 2, 3)

Assuming $t_o = \theta_o = 0$ and $\Lambda_i(t_o) \equiv \Lambda_{io}$, the zero-order equations are easily integrated to give

$$\begin{aligned}
\Lambda_1^o &= 3\Lambda_{4o}\theta/h_o + \Lambda_{1o} \\
\Lambda_2^o &= -2\Lambda_{4o}\sin\theta + \Lambda_{2o} \\
\Lambda_3^o &= 2\Lambda_{4o}(\cos\theta - 1) + \Lambda_{3o} \\
\Lambda_4^o &= \Lambda_{4o}
\end{aligned} \tag{5.29}$$

Note that these solutions also represent the circular coast-arc multiplier solutions.

Since the specified boundary conditions are satisfied by the zero-order solution, the boundary conditions for the first and higher-order perturbation equations are $\Lambda_i^j(0) = 0$ (i = 1,...,4; j = 1,2,...) .

To obtain the first-order solutions, the differential equation for Λ_4^1 must be solved first. This results in

$$\Lambda_4^1 = -\Lambda_{40}\theta + (\Lambda_{20} - \Lambda_{30}/2)\sin\theta + (1 - \cos\theta)\Lambda_{40}/2 \ . \tag{5.30}$$

After substituting Equation (5.30) into the remaining equations of (5.28), the first-order equations can then be solved by separation of variables. Thus, the solutions of Equations (5.25), to first-order in ε, are

$$\begin{aligned}
\Lambda_1 &= \Lambda_{10} + 3\Lambda_{40}\theta/h_o + \varepsilon\{(3\Lambda_{20}/h_o - \Lambda_{10})\theta/2 \\
&- 21\Lambda_{40}\theta^2/(8h_o) - 2\Lambda_{20}\sin\theta/h_o + \\
&+ (4\Lambda_{40} - 2\Lambda_{30})(1 - \cos\theta)/h_o\} \\
\Lambda_2 &= \Lambda_{20} - 2\Lambda_{40}\sin\theta + \varepsilon\{3\Lambda_{20}\theta/8 + \\
&+ (h_o\Lambda_{10}/4 - \Lambda_{20})\sin\theta + 9\Lambda_{40}(\cos\theta - 1)/4 \\
&+ (11\Lambda_{40}\theta\sin\theta)/4 - 5\Lambda_{20}(\sin\theta\cos\theta)/8 \\
&(5\Lambda_{30}/8 + 5\Lambda_{40}/4)\sin^2\theta\} \qquad\qquad (5.31) \\
\Lambda_3 &= \Lambda_{30} + 2\Lambda_{40}(\cos\theta - 1) + \varepsilon\{(-3\Lambda_{40}/4 + 3\Lambda_{30}/8)\theta \\
&+ (\Lambda_{20} - h_o\Lambda_{10}/4)(\cos\theta - 1) + 9\Lambda_{40}(\sin\theta)/4 \\
&- 11\Lambda_{40}(\theta\cos\theta)/4 + (5\Lambda_{40}/4 - 5\Lambda_{30}/8)\sin\theta\cos\theta \\
&+ 5\Lambda_{20}(\sin^2\theta)/8\} \\
\Lambda_4 &= \Lambda_{40} + \varepsilon\{-\Lambda_{40}\theta + (\Lambda_{40} - \Lambda_{30}/2)\sin\theta \\
&+ \Lambda_{20}(1 - \cos\theta)/2\}
\end{aligned}$$

Since these solutions contain secular terms, their range of applicability is not as great as the range of applicability for the state variable solutions. The secular terms are caused by the $(1 - \varepsilon\theta)^n$ - expansions, and for the solutions given above the expansions are about the point $\varepsilon\theta = 0$.

Since $0 \le \varepsilon\theta < 1$ on typical multirevolution trajectories of long duration, it is probable that the convergence can be improved by piecing together solutions obtained by using, for example, $\varepsilon\theta = 0$, .1 , .2 , ... , .9 as expansion points, instead of just $\varepsilon\theta = 0$. As shown in Chapter 7, the solutions of Equations (5.31) are good approximations to the true solutions for $\varepsilon\theta \le 0.25$.

CHAPTER 6

COMPUTATIONAL CONSIDERATIONS

In Chapter 7, the base canonic constant relationships developed in Chapters 3 and 4 are evaluated along several optimal circular orbit transfers and optimal escape trajectories. The generation of the optimal trajectories requires the numerical solution of a two-point boundary value problem. There exist various iteration schemes for solving these problems, and Reference 22 discusses the relative merits of several different iteration methods. Reference 22 also contains a representative Bibliography of the many technical papers written on this subject.

In the generation of the optimal transfer and escape trajectories, important trends, concerned with the methods for converging the trajectories, became apparent. For example, it was found that the little publicized secant method[43] was very effective in the convergence of escape trajectories, but it was not as effective as the linear perturbation method[22] in converging optimal circular orbit transfers. The apparent reasons for this difference are discussed in Sections (6.1) and (6.2).

6.1 Optimal Low-Thrust Escape Trajectories

In Section (3.1), necessary conditions for a Bolza problem in the calculus of variations were stated. For a time-optimal mission in which the only geometrical boundary condition at the final time is the requirement that the terminal energy satisfy the escape energy condition, the transversality conditions, expressed in polar coordinates, are

$$
\begin{aligned}
\psi_1 &\equiv H(t_f) - 1 = 0 \\
\psi_2 &\equiv [\lambda_1 - \lambda_3 r^2 u/k]_{t_f} = 0 \\
\psi_3 &\equiv [\lambda_2 - \lambda_3 r^2 v/k]_{t_f} = 0 \\
\psi_4 &\equiv \lambda_4(t_f) = 0 \, .
\end{aligned}
\tag{6.1}
$$

The escape condition will be denoted by

$$\psi_5 \equiv [u^2 + v^2 - 2k/r]_{t_f} = 0 \, . \tag{6.2}$$

The governing differential equations are Hamilton's equations for the following Hamiltonian

$$H = \lambda_1(v^2/r - k/r^2) - \lambda_2 uv/r + \lambda_3 u + \lambda_4 v/r + \frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2} \, . \tag{6.3}$$

The differential equations have three important properties:

(i) $\dot{\lambda}_4 = 0$, which implies λ_4 = constant $\equiv 0$ since $\lambda_4(t_f) = 0$;

(ii) the equations of motion only contain the Lagrange multipliers in the form $\lambda_1(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}}$ and $\lambda_2(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}}$;

(iii) the differential equations for the Lagrange multipliers form a linear system, say $\dot{\lambda} = g(x)\lambda$.

By property (i), the initial value of λ_4 is determined. Properties (ii) and (iii) imply that if ψ_2 , ψ_3 , ψ_4 , and ψ_5 are satisfied, then an optimal trajectory is defined. For example, suppose $\psi_2 = \psi_3 = \psi_4 = \psi_5 = 0$ and $\psi_1 = H(t_f) - 1 = C_o^*$, with $C_o^* \neq 0, -1$. Then, $H(t_f) = C_o^* + 1 \equiv \frac{1}{C_o}$. Let $\{\lambda_{10}^*, \lambda_{20}^*, \lambda_{30}^*, 0\}$ be the set of initial values for the Lagrange multipliers which generates this trajectory. Then, the set $\{C_o\lambda^*_{10}, C_o\lambda^*_{20}, C_o\lambda^*_{30}, 0\}$ generates the optimal trajectory for which $\psi_i = 0$ $(i = 1,\ldots,5)$. That is, since $\dot{\lambda} = g(x)\lambda$ is a linear system, the set $\{C_o\lambda_i(t)|i = 1,\ldots,4;$ $t_o \le t \le t_f\}$ can be formed by integrating $\frac{d}{dt}(C_o\lambda) = g(\dot{x})\ (C_o\lambda)$ from t_o to t. In particular, at t_f:

$$H(t_f;\ C_o\lambda) = C_o\ H(t_f;\ \lambda) = C_o(\frac{1}{C_o}) = 1$$

$$\psi_2' = [(C_o\lambda_1) - (C_o\lambda_3)r^2u/k]_{t_f} = C_o\ \psi_2(t_f) = 0$$

$$\psi_3' = [(C_o\lambda_2) - (C_o\lambda_3)r^2v/k]_{t_f} = C_o\psi_3(t_f) = 0$$

$$\psi_4' = (C_o\lambda_4)_{t_f} = C_o\ \lambda_4(t_f) = 0\ .$$

Clearly, the escape condition is still satisfied since

$$(C_o\lambda_1)\ [(C_o\lambda_1)^2 + (C_o\lambda_2)^2]^{-\frac{1}{2}} = \lambda_1[\lambda_1^2 + \lambda_2^2]^{-\frac{1}{2}}$$

$$(C_o\lambda_2)\ [(C_o\lambda_1)\ + (C_o\lambda_2)\ ]^{-\frac{1}{2}} = \lambda_2[\lambda_1 + \lambda_2]^{-\frac{1}{2}}\ .$$

That is, the state differential equations are unaffected when the Lagrange multipliers are scaled.

Since optimal low-thrust escape trajectories are reputed to be difficult to converge, the linear perturbation emthod was first employed to form the solutions. This method involves the integration of twice the number of given differential equations (i.e., the given equations and their variational equations), and also requires a guess for the final time of the trajectory. With very low-thrust, this method was not too successful.

Because of certain well-known characteristics of optimal escape trajectories with circular initial conditions, an alternate method which takes advantage of this knowledge was employed. This method, the secant method, is less complex than the linear perturbation method since: it only involves the numerical integration of the original set of differential equations, no final time estimate needs to be given, and it involves the guessing of only two initial Lagrange multipliers. The secant method proved to be very successful in the generation of very low-thrust optimal escape trajectories.

The procedure followed in applying the secant method to the escape problem will now be outlined. First of all, the iteration of five final conditions was reduced to a second-order iteration scheme by: (1) terminating the numerical integration when the escape condition was satisfied; (2) requiring $\lambda_4 \equiv 0$; and (3) using the previously discussed linear system property, which is equivalent to specification of one of the initial Lagrange multipliers.

Since it is well-known that optimal escape trajectories are closely approximated by circumferential thrust programs (i.e., $\alpha = 0$) for the initial portion of their trajectories, then

$$\tan \alpha_o = \frac{\lambda_{10}}{\lambda_{20}} \simeq 0. \tag{6.4}$$

Thus, the scaling condition is applied to λ_{20}, say $\lambda_{20} = 1$ for all iterates, and a logical first guess for λ_{10} is $\lambda_{10} = 0$. Since $\lambda_4 \equiv 0$, the only remaining multiplier to guess is λ_{30} .

As noted above, $\alpha = 0$ should be a good approximation for the initial portion of the trajectory. Thus, a good first guess should involve the assumption $\lambda_1(t) \simeq 0$. Consider the differential equation for λ_1:

$$\dot{\lambda}_1 \;=\; -\lambda_3 + \lambda_2 v/r \;.$$

To keep $\lambda_1(t) \simeq 0$, then $\dot{\lambda}_{10} \simeq 0$, so

$$-\lambda_{30} + \lambda_{20}\, v_o/r_o \approx 0 \;.$$

Since $\lambda_{20} = 1$ is assumed, an initial guess for λ_{30} is given by

$$\lambda_{30} \simeq v_o/r_o \;. \tag{6.5}$$

Thus,

$$\{\lambda_{10} \;=\; 0,\;\; \lambda_{20} \;=\; 1,\;\; \lambda_{30} \;=\; v_o/r_o,\;\; \lambda_{40} \;=\; 0\} \tag{6.6}$$

represents a set of first guesses for the initial Lagrange multipliers.

The first guesses of Equation (6.6) needed a slight modification since the trajectories "spiraled in" (instead of going out) with these guesses. The reason for this is apparent from the differential equation for $\dot{r}$, i.e.,

$$\frac{d\dot{r}}{dt} \;\equiv\; \frac{du}{dt} \;=\; v^2/r - k/r^2 + \frac{T}{m}\,\lambda_1(\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}} \;. \tag{6.7}$$

By requiring $\lambda_{10} = 0$, the radius developed a large oscillation. Intuition would lead one to believe that this is a nonoptimal effect. This large oscillation is due to the fact that the trajectory stays near-circular conditions (i. e., $v^2 = k/r$) for the initial portion of the trajectory and thus the $(v^2 - k/r)/r$ term in Equation (6.7), which is dominant since $\lambda_1(t) \simeq 0$, is oscillating about circular conditions. This condition can be averted by assuming a small, nonzero value for λ_{10} so that there will some interaction between the gravitational and the thrusting terms in Equation (6.7). Then the radial velocity will be small, and the radius will be a slowly-varying, well-behaved function. A good first guess for λ_{10} is a value which makes $\alpha_o \simeq \pm 0.5^o$. Table 1 presents the optimal initial multipliers and transit times for various values of thrust.

For some missions, the final-value of the range angle, θ, is important. The analysis developed above is still applicable to such missions if the initial-value of the range angle is unimportant. For example, suppose θ_c is the desired final-value of θ, and $\theta(t_f) \neq \theta_c$ is the final-value of θ on an optimal trajectory with $\theta(t_o) = \theta_o$. Then, the desired optimal trajectory can be determined by a rotation in θ defined by $\Delta\theta = \theta_c - \theta(t_f)$. Thus, for this case, the initial-value of θ will now be $\theta_o + \theta_c - \theta(t_f)$.

An indication of the accuracy of the numerical integration scheme can be obtained by adjoining a multiplier for the mass to the Hamiltonian, i.e.,

$$H^* = H + \lambda_5 \dot{m} , \tag{6.8}$$

and numerically integrating the differential equation for λ_5. Then, since time does not appear explicitly in the new Hamiltonian, H^*, it will be a constant of the motion. The number of digits which H^* holds over the entire trajectory gives an indication of the accuracy of the numerical integration procedure.

Concluding this section, it appears that the reason for the success of the secant method was due to the relative weakness of the terminal geometrical constraints on the problem. That is, the only terminal geometrical boundary condition was the escape condition, i.e., $u_f^2 + v_f^2 - 2k/r_f = 0$. In contrast, the secant method was ineffective when applied to the problem of the next section, which involves three terminal geometrical constraints.

6.2 Optimal Circular Orbit Transfers

The only difference between the formulations for the optimal escape trajectory and the optimal circular orbit transfer trajectories is the specification of terminal boundary conditions and the resultant transversality conditions. The terminal geometrical boundary conditions for a circular orbit transfer are

$$\begin{aligned} \psi_1 &\equiv r(t_f) - r_c = 0 \\ \psi_2 &\equiv u(t_f) = 0 \\ \psi_3 &\equiv v(t_f) - (k/r_c)^{\frac{1}{2}} = 0 \; . \end{aligned} \tag{6.9}$$

The remaining terminal conditions are given by the transversality conditions

$$\begin{aligned} \psi_4 &\equiv H(t_f) - 1 = 0 \\ \psi_5 &\equiv \lambda_4(t_f) = 0 \; . \end{aligned} \tag{6.10}$$

The governing differential equations are the same as in Section 6.1 and since $\lambda_4 \equiv 0$, then $\lambda_{40} = 0$ is defined for the iteration scheme.

As a result of the success of the secant method in determining the optimal escape trajectories, the secant method was applied to the optimal transfer problem. It was found that when ψ_2 (of Eqs. (6.9)) was not near zero on an iterate, the secant method usually would not converge the trajectory. Thus, the linear perturbation method was employed. This method proved to be more effective than the secant method, but judicious guesses for the initial multipliers and final time were necessary for convergence.

As a first guess, the $\alpha(t) \approx 0$ assumption was used to guess the initial multipliers. After a few unsuccessful attempts, the ten pound thrust case was finally converged. Then, it was a relatively simple manner to step down (in increments of one pound of thrust) to the five pound case. Both the escape and transfer analyses were determined only for thrust forces ranging from ten- to five-pounds since the five-pound trajectories were already requiring in the neighborhood of one million seconds, real time.

In all of the circular orbit transfers, effective use was made of the approximate solutions of Chapter 5. The second-order approximation of the final time, which was determined by a simple Newton's iteration on Equations (5.5), was always within 1.0% of the optimal transfer time. This aided the convergence properties of the linear perturbation method considerably. Table 2 lists the optimal initial multipliers and the transit times (along with the approximate transit times given by the analysis of Chapter 5) for several values of thrust and specific impulse.

CHAPTER 7

QUALITATIVE ASPECTS OF OPTIMAL LOW-THRUST TRAJECTORIES

In this chapter certain qualitative aspects of the solutions formed in Chapters 3, 4, and 5 will be discussed. The canonic variables which were developed in Sections (3.3) and (4.2) are evaluated along representative optimal trajectories. The resultant trends are discussed in Sections (7.1) and (7.2) and displayed graphically in Figures 4 through 13. These evaluations also give an indication of the functional form of the canonic perturbation equations without performing the tedious development of the equations.

In Section (7.3) the non-Hamiltonian solution, which was developed in Chapter 5, is discussed. Figures 14 through 25 compare the approximate solution with the true circumferential and true optimal solutions. This analysis verifies the conjecture that the circumferential states and multipliers are good approximations to their optimal counterparts in multirevolution, low-thrust missions.

7.1 The Canonic Constants of the Coast-Arc Problem

In this section the canonic variables defined by Eqs. (3.67), (3.69), and (3.72) will be discussed. First note that the first equation of Eqs. (3.72), i.e.,

$$\beta_1 = q_4 \tag{7.1}$$

where $q_4 \equiv \theta$, is not a constant of the motion for the coast-arc. The reason for the nonconstant character of β_1 was discussed previously (see page 57). Since the Hamiltonian for the coast-arc problem in the $\{\alpha, \beta\}$-system is just $H_o = \alpha_1$, then

$$\frac{d\beta_1}{d\theta} = \frac{\partial H}{\partial \alpha_1} = 1 \rightarrow \beta_1 = \theta + b_1 . \tag{7.2}$$

Thus, a canonic constant, b_1, can be easily formed from Eq. (7.1) and the result is trivial, i.e., $b_1 \equiv 0$. The triviality of this result is a consequence of the artifice involved in treating the independent parameter (θ) as a coordinate.

As noted in Section (3.4), the set $\{\alpha_1,\ldots,\alpha_5\}$ is well-defined for both the hyperbolic and parabolic cases as well as for the elliptic case. Thus, singularity difficulties are not encountered with these variables. The time rates of change of these variables for representative cases are displayed in Figs. 4 through 7. The time rate of change of α_4 $(\equiv e \sin\omega)$ is not shown since its behavior is similar to α_3 $(\equiv e \cos\omega)$.

Figure 4 exhibits the behavior of α_1, the base Hamiltonian, along the optimal five-pound-thrust transfer. The behavior along the optimal five-pound-thrust escape is similar to α_1 of Fig. 4 for the first 9×10^5 seconds of the trajectory, and then α_1 becomes a monotonically decreasing function until the end of the trajectory (reaching a minimum value of -160,000). This change in character of α_1 on the escape trajectory is probably due to the reduction in strength of the gravity force, with respect to the thrust force, as the radius increases. Near the escape condition the gravity and thrust forces are of the same order of magnitude, so the thrust is no longer a small-parameter. Although α_1 is an oscillatory function, the mean value of the function is very slowly varying, and the amplitude of the oscillation is relatively small for the major portion of the trajectory.

The angular momentum variable, α_2 , (actually, α_2 differs from the classical angular momentum by $k^{-\frac{1}{2}}$) is displayed in Figure 5. In this case the ten-pound-thrust case is shown also. This demonstrates the expected result that a higher thrust level causes a more rapidly varying variable. However, the functional forms of the two cases are almost identical, and this property is true for the other variables as well. Thus, the ten-pound-thrust case is only presented when the clarity of presentation is not affected. In both cases α_2 is a monotonically increasing, nonoscillatory variable.

A partial time history of α_3 for the circular oribt transfer is shown in Fig. 6. In Figs. 16 and 17 the dashed-curves represent the complete time histories of α_3 for both the transfer and escape trajectories. Although α_3 remains small for the entire transfer and for the major portion of the escape trajectory, it is an oscillatory function with monotonically increasing amplitude and period. The increase in period is due to the fact that as the radial distance of the vehicle increases, a complete revolution about the central body takes a longer amount of time. Thus, the instantaneous orbital parameters become more slowly varying, and in particular, the argument of perihelion (which is the major cause of the rapid oscillation) does not change as fast. As previously mentioned, the qualitative behavior of α_4 is similar to that of α_3 .

In Fig. 7 the total time histories of α_5 are shown. These slowly-varying, near-linear curves indicate the validity of the constant mass assumption in low-thrust analyses. That is, since $\alpha_5 \equiv \Lambda_5$ and

$$\frac{d\Lambda_5}{dt} = \frac{-\partial H}{\partial t} = -\frac{\partial H}{\partial m}\frac{\partial m}{\partial t} = -\dot{m}_o\frac{\partial H}{\partial m} ,$$

if $\dot{m}_o$ is small, then Λ_5 should be slowly varying. Also note that $\alpha_5(t_f) = -1$ for each trajectory. This is a consequence of a transversality condition for variational problems in which the final time is unspecified (e. g., see the first equation of Eqs. (6.1) and Eqs. (6.10)).

At the escape condition, the instantaneous state of the vehicle defines a parabolic orbit, i.e., unit eccentricity. Thus,

$$\alpha_3^2 + \alpha_4^2 = e^2\cos^2\omega + e^2\sin^2\omega = e^2 = 1 .$$

Therefore,

$$1 - \alpha_3^2 - \alpha_4^2 = 0 . \qquad \text{(At escape.)}$$

Insepction of the $\{\beta_1,\ldots,\beta_5\}$ - set, defined by Eqs. (3.72), shows that the β_2 , β_3 , β_4 , and β_5 equations contain $(1 - \alpha_3^2 - \alpha_4^2)$ - terms in the denominators of the right hand sides of their equations. Thus, the $\{\beta_i\}$ set must be restricted to either elliptic, parabolic, or hyperbolic conditions.

A partial time history of β_2 for the circular orbit transfer is presented in Fig. 8. Λ_1 appears linearly in the right-hand side of the β_2-expression. Thus, β_2 is "generated" by Λ_1 . Figures 20 and 21 show that Λ_1 is a monotonically decreasing funciton for the major portion of both the transfer and escape trajectory, and Fig. 8 shows that β_2 is a monotonically increasing function. Contrary to intuition, Λ_1 is a more slowly-varying function than β_2 . That is, it appears that since the differential equations for Λ_1 and β_2 are of the general form

$$\begin{aligned} \frac{d\Lambda_1}{d\theta} &= \frac{T}{m}\tilde{f}(X, \Lambda) + \tilde{g}(k, X, \Lambda) \\ \frac{d\beta_2}{d\theta} &= \frac{T}{m} f^*(\alpha, \beta) , \end{aligned} \qquad (7.3)$$

then, in analogy with the state equations, intuition would lead one to believe that β_2 should be more slowly varying than Λ_1 since the <u>total</u> right-hand side of its differential equation is multiplied by the small-parameter $\frac{T}{m}$. However, the right-hand sides of Eqs. (7.3) do not represent forces/unit mass and, thus, the same argument does not apply. For example, near the midpoint of the five-pound-transfer

$$\frac{d\Lambda_1}{d\theta} \approx \{(6 \times 10^{-3})\Lambda_1 - 30(\Lambda_2\cos\theta + \Lambda_3\sin\theta)\} + (1.5 \times 10^4)\Lambda_4 , \qquad (7.4)$$

where $\{\cdots\}$ contains $\frac{T}{m}$ and $(1.5 \times 10^4)\Lambda_4$ does not. From Figs. 20, 22, and 24, $\Lambda_1 \approx 5 \times 10^9$, $\Lambda_2 \approx 2 \times 10^4$, and $\Lambda_4 \approx \pm 500$. Since Λ_3 is of the same order of magnitude as Λ_2, Eq. (7.4) becomes

$$\frac{d\Lambda_1}{d\theta} \approx \{3 \times 10^7 \pm 6 \times 10^5\} \pm 8 \times 10^6 .$$

Thus, the thrust portion is actually greater than the nonthrust protion of the equation. Also, since the nonthrust portion can be negative, the total rate of change may be less than the thrust rate of change alone. This heuristic argument accounts for the more rapid variance of β_2 (as opposed to Λ_1).

The third equation of Eqs. (3.72) shows that Λ_2 "generates" β_3, and the time histories of these variables are presented in Figs. 9, 22, and 23. Both Λ_2 and β_3 are oscillatory functions with increasing periods of oscillation. For the first half of both trajectories, the mean-value of β_3 varies more slowly than the mean-value of Λ_2. This is probably due to the fact that β_3 is a canonic constant for the coast-arc defined by the initial conditions, and, thus, β_3 stays near $\beta_3(t_o)$ until the character of the trajectory begins to change (i.e., deviate from very-near-circular conditions).

Then, for the second half of both trajectories, the amplitude of oscillation of β_3 becomes greater than the amplitude of oscillation of Λ_2, as was the case with Λ_1 and β_2 mentioned above.

In Fig. 10, a partial time history of β_5 (the canonic variable generated by t) is exhibited. Note that for the first 200,000 seconds β_5 is more slowly varying than t (i.e., $-t$ vs. t is represented by the dashed curve for comparison). This is because the state of the vehicle is still very near circular conditions. This fact can be verified by considering the β_5 - relation

$$\beta_5 = t - h^3 k\theta + O(e) ,$$

and the $t(\theta)$ relation for a circular orbit, i.e.,

$$t = h^3 k\theta . \qquad \text{(circular conditions; } t_o = \theta_o = 0\text{)}$$

That is, β_5 is essentially "generated" by the circular $t(\theta)$ relationship, so it is slowly varying when the eccentricity, e, is very small (i.e., near-circular conditions).

Concluding this section, the $\{\alpha, \beta\}$ - set represents a well-defined set of variables for missions with instantaneous elliptical (including circular) conditions. The differential equations in the $\{\alpha, \beta\}$ - system for any space mission involving an inverse-square gravity field and a thrusting force will be of the form

$$\frac{d\alpha}{d\theta} = \frac{T}{m} \tilde{f}(\alpha, \beta)$$

$$\frac{d\beta}{d\theta} = \frac{T}{m} f^*(\alpha, \beta) .$$

As discussed above, the new state related variables will be more slowly varying than the original state variables if $\frac{T}{m}$ is small, whereas the new multiplier related variables do not necessarily possess this characteristic.

7.2 The Canonic Constants of the Circumferential Thrust Problem

Before the presentation of the time variations along the optimal trajectories of the canonic variables defined by the Hamilton-Jacobi solution in Section (4.2), a brief discussion about the general assumptions used in determining the solution will be given. Recall that three basic assumptions were made for the trajectories of interest: (i) the magnitude of the control angle, α , is less than 45°; (ii) $1 > \varepsilon\theta$ for each $\theta \in [\theta_o, \theta_f]$; and (iii) the instantaneous eccentricity is small for the total optimal transfer and for the major portion of the optimal escape trajectory.

In Fig. 11 the optimal control angle histories for four representative missions are shown. On each of these missions the control is an oscillatory function with increasing period and amplitude. On the optimal transfers the magnitude of the control is never greater than 35°, and a lower thrust level implies a smaller maximum value for the control (i.e., a more nearly circumferential thrust orientation). The magnitude of the control on the optimal escapes stays less than 10° for the major portion of the trajectories. The sharp "swing-out" of the control near the escape condition is due to the tendency of the optimal trajectory to go to radial thrust at escape. The probable reason that the escape program is closer than the transfer program to circumferential thrust is that the geometrical constraints on the transfer are more rigid. That is, a greater number of geometrical constraints on the problem usually implies a more complex optimal control law.

The maximum values of the range angle, θ , on the optimal trajectories are shown in Figs. 12a and 12b. In each of these cases, the inequality $1 > \varepsilon\theta$ is satisfied. Since $\varepsilon\theta$ approaches 1 on each of these missions, a good approximation for the number of revolutions, n , that a multirevolution low-thrust trajectory takes is

$$n \approx \frac{1}{2\pi\varepsilon} \tag{7.5}$$

Figures 16 and 17 indicate the eccentricity values along typical optimal trajectories since $q \equiv e \cos\omega$. On the optimal transfer, $e < 0.1$ for the entire trajectory; and on the optimal escape, $e < 0.1$ for approximately 80% of the trajectory.

The time rates of change of the $\{A, B\}$ - set, developed in Sec. (4.2), along optimal trajectories will now be discussed. First, consider Fig. 4 which exhibits a partial time history of A_1 (the base Hamiltonian) for the five-pound-thrust transfer. The character of A_1 on the five-pound-thrust escape is similar, and A_1 becomes unbounded at escape since it depends on β_i's which are undefined at escape. The scale for A_1 is ten times greater than the scale for α_1 . Thus, for the first 400,000 seconds of the trajectory A_1 deviates only slightly away from the base value (i.e., $A_1(t_o)$). When compared with α_1 , both curves are oscillatory with similar periods of oscillation. However, since the scale for A_1 is ten times greater than the scale for α_1 , the mean-value of α_1 varies more rapidly than the mean-value of A_1, and the amplitude of oscillation of α_1 is greater than the amplitude of oscillation of A_1 . Near the end of the trajectory, the amplitude of A_1 becomes larger than the amplitude of α_1 .

In Fig. 5 the total time histories of A_5 for both the five-and ten-pound cases are presented. These curves show that A_5 is more slowly varying than its comparable variable, α_2 . Furthermore, for the transfer, A_5 is nearly constant. This fact may be of great use in future analytic investigations of optimal circular orbit transfers.

Since $A_2 = \beta_3$, $A_3 = \beta_4$, and $A_4 = \alpha_5$, these variables will not be discussed here. For a discussion of these variables, see the previous section. Also, the Hamilton-Jacobi solution of Sec. (4.2) results in the equilibrium solution so that

$$B_1 = \theta - Q_1 = \theta - \theta \equiv 0$$

gives a trivial relationship.

Each of the variables B_2 , B_3 , B_4 , and B_5 depends upon an approximate evaluation of the indefinite integral

$$\int \left(\frac{\partial S^*}{\partial Q_1} \right)' dQ_1 , \tag{7.6}$$

defined by Eq. (4.41). The evaluations represented in Figs. 6, 8, and 10 were determined by integrating Eq. (7.6) after expanding the integrand about $\epsilon\theta = 0$. The B_i's were then evaluated (keeping up to second-order terms in ϵ). Thus, the degree of improvement of the B_i's over their counterparts in the $\{\alpha, \beta\}$ - set depends upon how well Eq. (7.6) can be approximated. Since the expansions for Figs. 6, 8, and 10 are about $\epsilon\theta = 0$, the best comparison should be in the neighborhood of $t = 0$.

First consider Fig. 6. The time history of B_2 shows a large improvement over α_3 in the neighborhood of $t = 0$. Note that the undesirable rapid oscillation of α_3 is not present in B_2 . The oscillation in B_2 which begins at approximately 75,000 seconds could probably be overcome by piecing together expansions about $\epsilon\theta = 0$, .1 , .2 , etc. The behavior of B_3 is essentially the same as B_2 .

In Fig. 10 the time history of B_4 is presented. In a neighborhood of $t = 0$ (approximately $t \epsilon [0, 10^5]$) B_4 is essentially constant. Then as the effects of the approximation of Eq. (7.6) begin to appear, the amplitude of oscillation of B_4 increases. In Sec. (4.2) the relationship between b_4 (of Sec. (4.1)) and B_4 was discussed. The time histories of b_4 are given in Fig. 13. Note that b_4 is a slower-varying, smoother function than B_4 . This is due to the fact that B_4 depends upon an integral approximation whereas b_4

is known in closed-form (recall the discussion at the end of Sec. (4.2)). If Eq. (7.6) could be evaluated in closed-form, then B_4 should be even more slowly-varying than b_4 since B_4 represents a greater portion of the total Hamiltonian (i.e., compare the Hamiltonians of Eq. (4.5) and Eq. (4.26)).

Finally, consider Fig. 8, which contains a partial time history of B_5. In the neighborhood $t \in [0, 3.5 \times 10^4]$ of $t = 0$, B_5 is more slowly-varying than β_2. Then, however, B_5 develops a large oscillation. Inspection of the last of Eqs. (4.44) shows that B_5 is affected more than any of the other B_i's by the $\varepsilon\theta$ - expansion. This is the most probable reason for its large, rapid oscillation.

7.3 The Non-Hamiltonian Approximate Solution

In Figs. 14 through 25 the complete time histories of the true optimal, true circumferential, and circumferential approximation (of Chapter 5) for both the five pound transfer and escape trajectories are shown. First, recall from Chapter 5:

(a) $h(\theta)$ and $t(\theta)$ are asymptotic solutions with the single parameter $\delta \equiv \frac{m_o}{m_o}$, and they are taken to second-order in δ ;

(b) $q(\theta)$ and $s(\theta)$ are taken to second-order in δ, also; however, their components are functions of $\varepsilon\theta$ - expansions, and they are taken to third-order in ε ;

(c) the $\Lambda_i(\theta)$ are taken to first-order in ε, and their solution requires the constant-mass assumption.

In Figs. 14 and 15 the time histories of h are shown. For both cases, the true optimal and true circumferential values of h are nearly equal. The circumferential approximation is exceptionally good for the transfer and for the first two-thirds of the escape. These two cases demonstrate the dependence of the approximation on the small-eccentricity assumption. That is, on the transfer the eccentricity is always small and the approximation is very good for the entire trajectory; whereas, the circumferential approximation for the escape begins to deteriorate as the escape condition is approached (i.e., as unit eccentricity is approached).

The time histories of q are shown in Figs. 16 and 17. The approximations to the true circumferential value are fairly good for the interval $t \varepsilon [0, 3 \times 10^5]$. After this interval, the amplitude of the oscillation levels off, whereas the true circumferential amplitude continues to grow. This is probably due to the fact that the q-approximation is dependent upon an $\varepsilon\theta = 0$ expansion point. Thus, if expansions about other points are pieced together, then an improvement in the approximation may result. Also note that the q-value for the optimal escape is more nearly circumferential than the optimal transfer. This concurs with the discussion of Fig. 4 at the beginning of Sec. (7.2).

In Figs. 18 and 19 the time-range angle relations are given. All three curves are nearly equal for each case. However, as Fig. 19 shows, the escape case is not quite as good as the transfer. Thus, for the transfer, the optimal histories for $h(t)$ and $t(\theta)$ are nearly circumferential, and the circumferential approximations of Chapter 5 are very good.

The Lagrange multiplier histories for the circumferential cases were evaluated by using the initial conditions of the true optimal trajectories. In Figs. 20 and 21 the time histories of Λ_1 are presented, and the true circumferential and true optimal cases give good agreement. They only begin to differ in a small neighborhood of the terminal point. Considering the assumptions involved in forming the circumferential approximation, it is a relatively good approximation. It possesses the functional form of the true solution, and by considering more than one expansion point for the $(1 - \varepsilon\theta)^n$ expressions, further gains in accuracy can be expected. Also, as shown in Chapter 5, the higher order terms (in ε) can be obtained in closed-form, but this introduces quite a bookkeeping problem.

In Figs. 22 and 23 the time histories of Λ_2 are presented. The true circumferential and true optimal solutions are not as close as the corresponding Λ_1 solutions. However, they have similar functional forms and it appears that the true circumferential solution would only need a slight perturbing decrease in value to duplicate the true optimal solution. The circumferential approximation is good for approximately the first 10^5 seconds, and then it levels off. The solutions for Λ_3 have the same character as the solutions for Λ_2 .

The total time histories of Λ_4 are presented in Figs. 24 and 25. The true circumferential solution closely approximates the true optimal solution for approximately the first two-thirds of each trajectory. As with Λ_2 , even though the two solutions begin to diverge on the final portion of the trajectory, they still have similar functional forms. Again the circumferential approximation is good for approximately the first 10^5 seconds, and then it levels off to a near constant amplitude.

Certain characteristics are evident for all of the Λ_i solutions. First, the periods of the oscillations for the three solutions (i.e., true optimal, true circumferential, and approximate circumferential) are nearly the same. For example, consider Fig. 22. Each of the three solutions have their hills and valleys at approximately the same time. Thus, only the secular (long-term) effects need to be better approximated in the circumferential approximation. This suggests a future study of the problem which makes use of the method of averaging.

Probably a more important characteristic is the fact that the optimal control is a parameter which does not greatly affect the functional form of the circumferential state or multiplier solutions of the given differential equations. Thus, one can be more confident that a perturbation analysis of the circumferential solution will lead to good approximations of the true optimal behavior.

The analysis of this section shows that the conjecture brought forward in Chapter 4 (i.e., if the optimal state is near the circumferential state, then the optimal multipliers are near the circumferential multipliers) is a valid hypothesis. This result is clear for the major portion of the escape mission since the circumferential state variables are close to the optimal state variable. However, by Fig. 16, the optimal q (and thus, s) is not closely approximated by the circumferential q for the circular orbit transfer, and yet the two sets of Lagrange multipliers are relatively close. This apparent discrepancy is resolved by the fact that q and s are small for the entire trajectory and, thus, do not exert a great influence on the functional form of the multipliers. Hence, h and θ (or t), which are near-circumferential on the optimal transfer, essentially determine the functional form of the Lagrange multiplier solutions.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Summary

In the previous chapters, the methods of classical Hamiltonian mechanics are applied to the optimal trajectory problem. The adaptation and application of general canonical transformation theory to the optimal trajectory problem is described. Furthermore, nonsingular Hamilton-Jacobi solutions of the coast-arc and approximate circumferential thrust problems, defined by generalized Hamiltonian functions, are obtained. These solutions are represented by canonic constants so they define natural base solutions for further canonic perturbation analyses. The time rates of change of these canonic constants are evaluated along representative optimal low thrust escape and circular orbit transfer trajectories. These evaluations indicate qualitative characteristics of the motion of the optimal low-thrust trajectories of interest.

8.2 Conclusions

1. The applicability of general canonical transformation theory in optimal trajectory analysis has been demonstrated. Also, important applications of the subgroup of homogeneous canonical transformations were brought forward in Chapter 2.

2. The Hamilton-Jacobi canonical transformation was applied to a number of problems. In Sec. (3.2) a nonseparable H-J equation was solved by using a combination of the separation of variables method and the method of characteristics. In Sec. (3.3) the solution of Sec. (3.2) was improved by using a non-Hamilton-Jacobi transformation in conjunction with a new H-J

transformation. Both of these solutions determine sets of canonic constants for the coast-arc problem. The solution of Sec. (3.3) represents a new, nonsingular (at circular conditions) solution of the coast-arc problem.

In Chapter 4 two H-J solutions of the circumferential thrust problem were formed. These solutions are valid for low-thrust circular orbit transfers and the major portion of low-thrust escape trajectories with circular initial conditions. These solutions represent a sound foundation for future analytic investigations concerned with obtaining optimal feedback guidance functions since the stopping points of these analyses are well-defined by the Hamiltonians of Eqs. (4.5) and (4.26). Any perturbation of these solutions which includes the B-term (defined by Eq. (3.54)) in the new Hamiltonian should lead to a good approximate feedback guidance function.

3. Non-Hamiltonian solutions of the circumferential states and multipliers were formed in a straightforward manner in Chapter 5. The circumferential approximation of the state is exceptionally good as long as the eccentricity is relatively small, and the approximate multipliers are valid in a relatively large neighborhood of their expansion point (i.e., $0 \le \varepsilon\theta < .25$). Furthermore, these solutions may be improved by either carrying higher-order terms or piecing together expansions about different points (e.g., $\varepsilon\theta = 0 , .1 , .2 , \ldots$).

Consideration of the circumferential solution indicated the following characteristics of the optimal low-thrust trajectories under consideration: (i) the circular orbit transfer has a small eccentricity value for the entire trajectory, and the escape trajectory has a small eccentricity value for approximately 80% of the trajectory; (ii) the control angle on the first 80% of the escape lies in a smaller neighborhood of circumferential control than the corresponding neighborhood for the total transfer; and (iii) the optimal control does not greatly affect the functional form of the solution when it is considered as a parameter which perturbs the circumferential solutions into the true optimal solutions.

4. The conjecture: "If the optimal state is approximately circumferential, then the optimal multipliers can be approximated by the circumferential multipliers" was introduced, and validated numerically for several cases in Chapter 7.

5. The following property was observed: if an optimal trajectory problem is in the form

$$\begin{aligned} \dot{x} &= \tilde{g} + \varepsilon \tilde{f} \\ \dot{\lambda} &= g^* + \varepsilon f^* \end{aligned} \qquad (\varepsilon \text{ is small})$$

then a transformation $\{X(x, \lambda), \Lambda(x, \lambda)\}$ which results in

$$\begin{aligned} \dot{X} &= \varepsilon \tilde{F} \\ \dot{\Lambda} &= \varepsilon F^* \end{aligned}$$

does not necessarily imply that the Λ_i's vary more slowly than the λ_i's. However, the X_i's will usually be slower-varying than the x_i's since the right-hand sides of the $\dot{x}_i$ - equations represent known accelerations.

6. If the Poincare variables (and their associated multipliers) are used in a numerical study of the low-thrust problem, then a variable time-step numerical integration scheme should be used for greatest efficiency. That is, in Chapter 7 it was shown that the period of the oscillatory variables increases so the step size should be a function of the instantaneous period.

7. A simple method for approximating the neighborhood of the initial Lagrange multipliers for the numerical iteration of low-thrust two-point boundary value problems was presented in Chapter 6. Also, it was shown that the secant method can be very useful in the numerical solution of the two-point boundary problem associated with the optimal escape problem. It appears that other trajectory problems of similar character (i.e., nonrigid terminal geometrical constraints) may be iterated most efficiently by the secant method, also.

8.3 Recommendations

1. The remaining Hamiltonian of Eq. (4.26) should be attacked by studying the time rates of change of the $\{A, B\}$ - set in order to determine new simplifying assumptions. Also, the first-order eccentricity terms should be studied, and the constant mass assumption should be relaxed. The results of Chapter 5 can be used as a guide in obtaining the variable mass solutions.

2. Similar solutions should be obtained for problems which do not have small eccentricity values or small control angle values.

3. The problem should be reformulated in universal variables and an attempt should be made to obtain a uniformly valid set of canonic constants for the coast-arc problem.

4. Use should be made of the fact that the optimal multipliers have the same functional form as the circumferential multipliers. This fact may be useful in either numerical or analytical investigations of the problem.

5. The approximate multiplier solutions of Chapter 5 should be extended by considering variable mass, higher-order terms in ε , and piecing together the solutions obtained by using more than one expansion point.

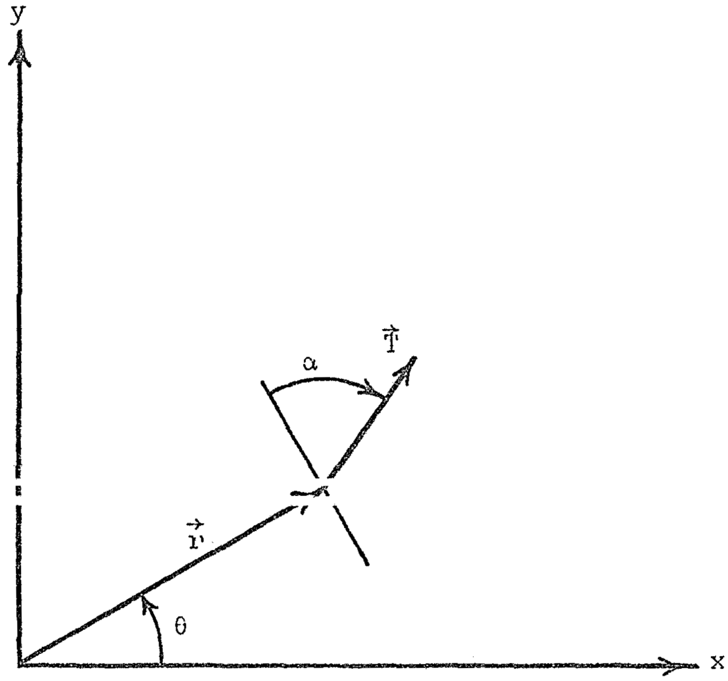


FIGURE 1. Coordinate System and Control Angle Definition

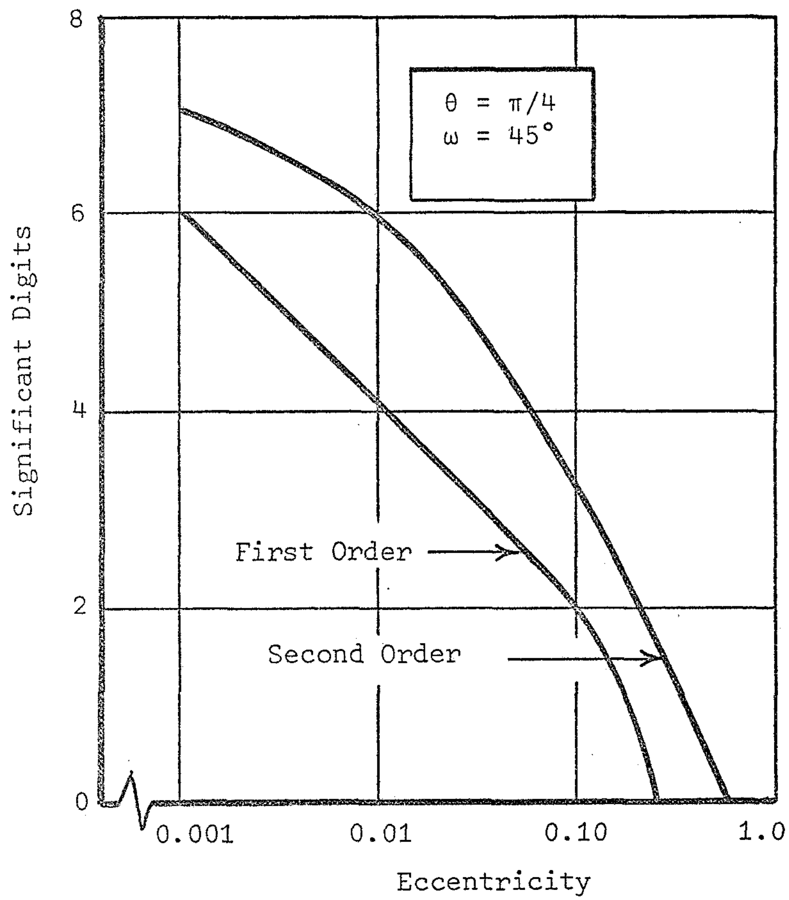


FIGURE 2. Validity of Y_2-Approximation for a Small Polar Angle

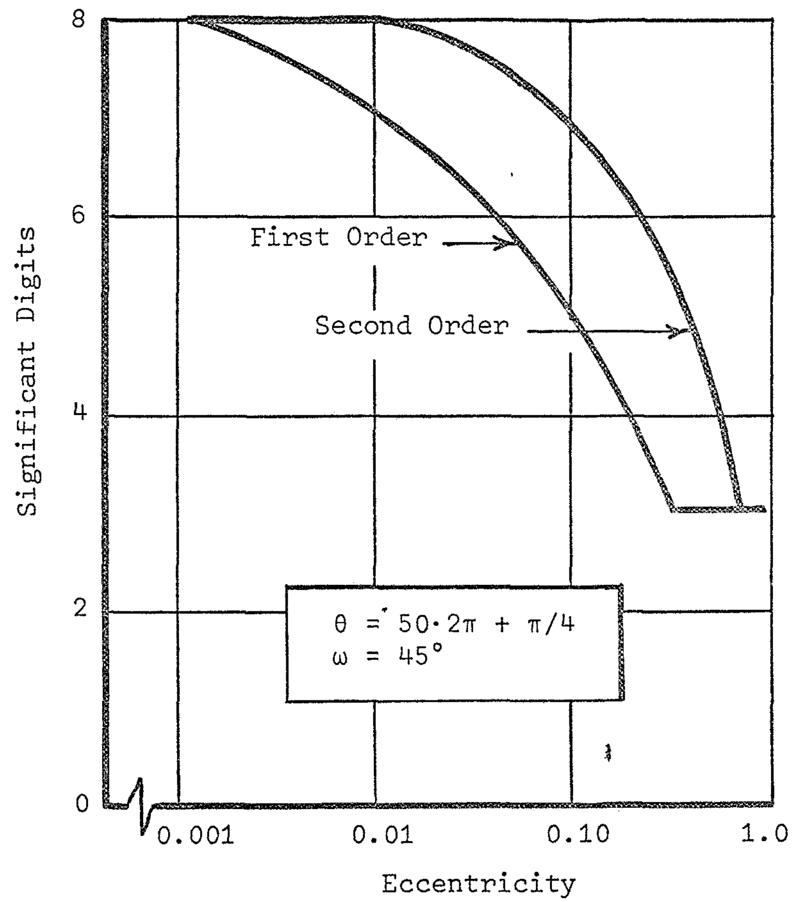


FIGURE 3. Validity of Y_2-Approximation for a Large Polar Angle (After 50 Revolutions)

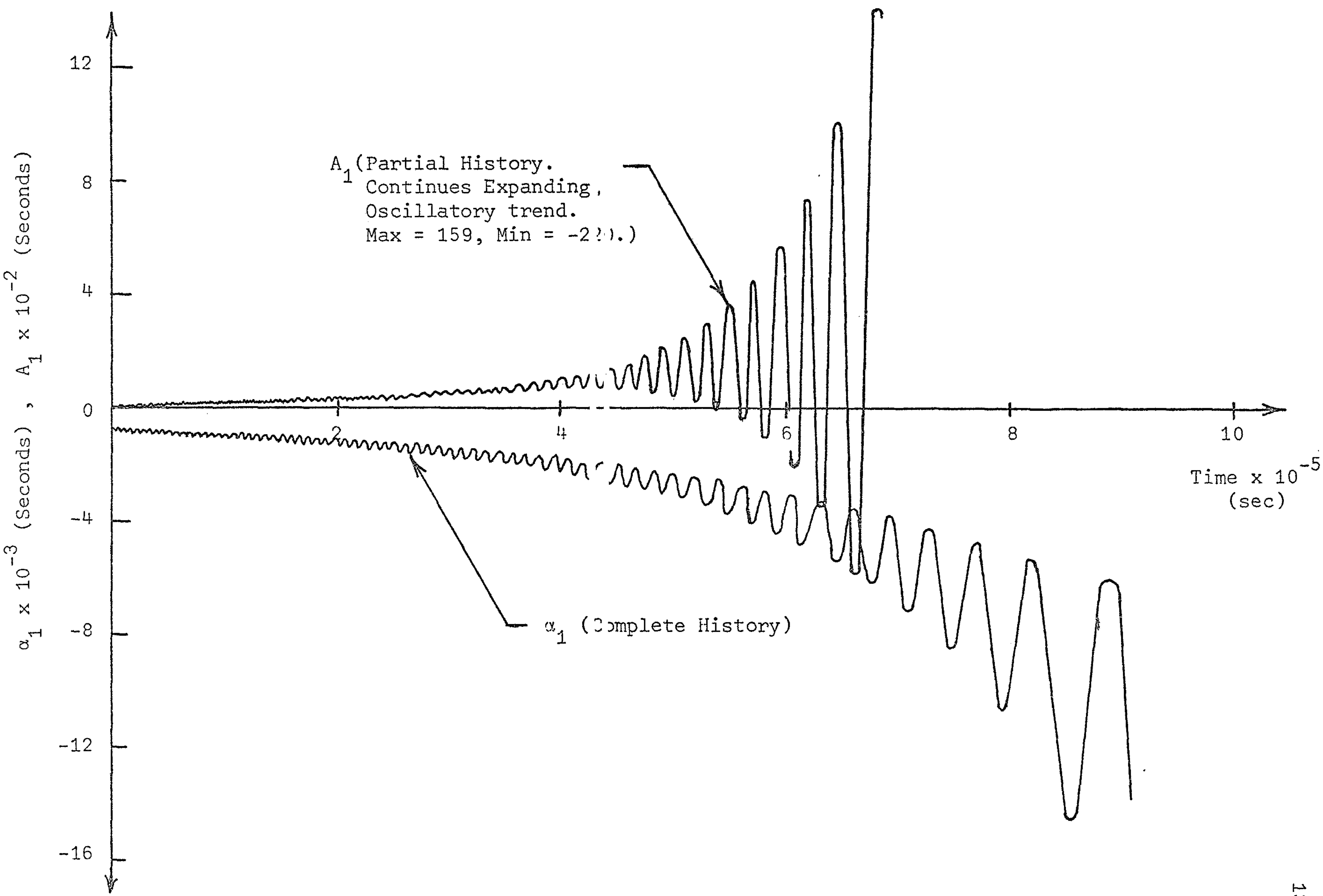


FIGURE 4. Time Histories of the Canonic Variables Generated By the Base Hamiltonians (5#-Transfer)

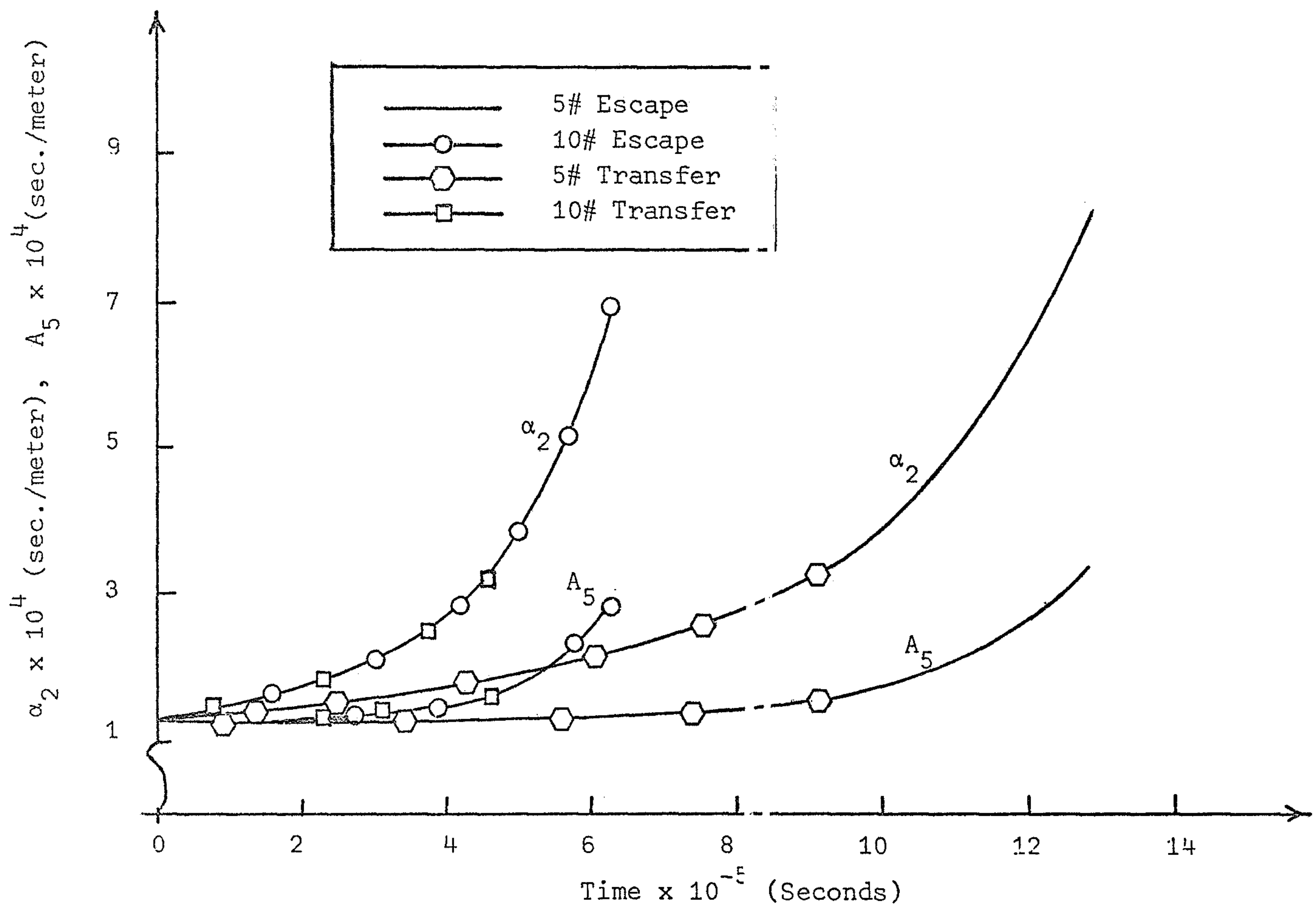


FIGURE 5. Time Histories of The Canonic Variables Generated by h

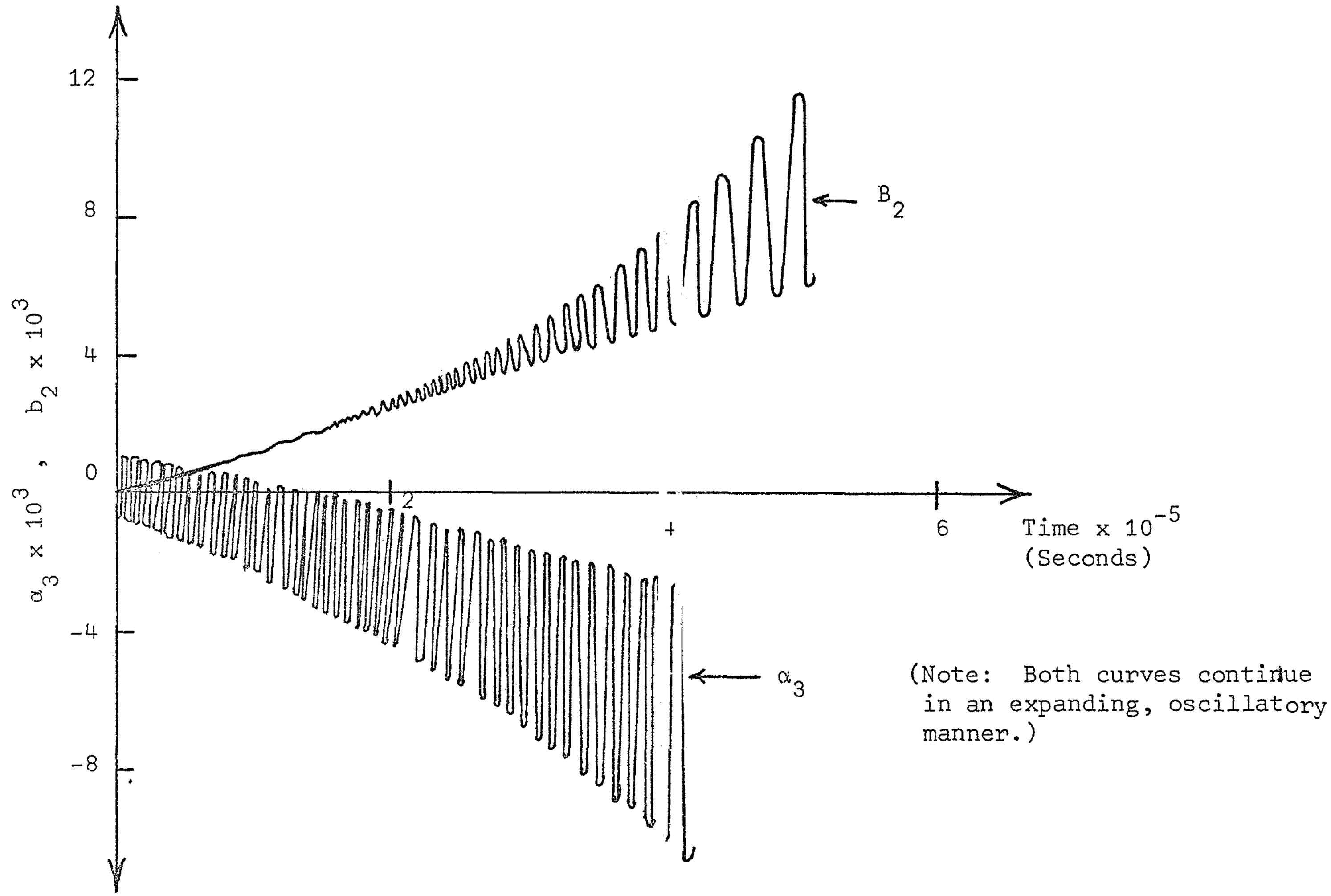


FIGURE 6. Partial Time Histories of The Canonic Variables Generated By q(5#-Transfer)

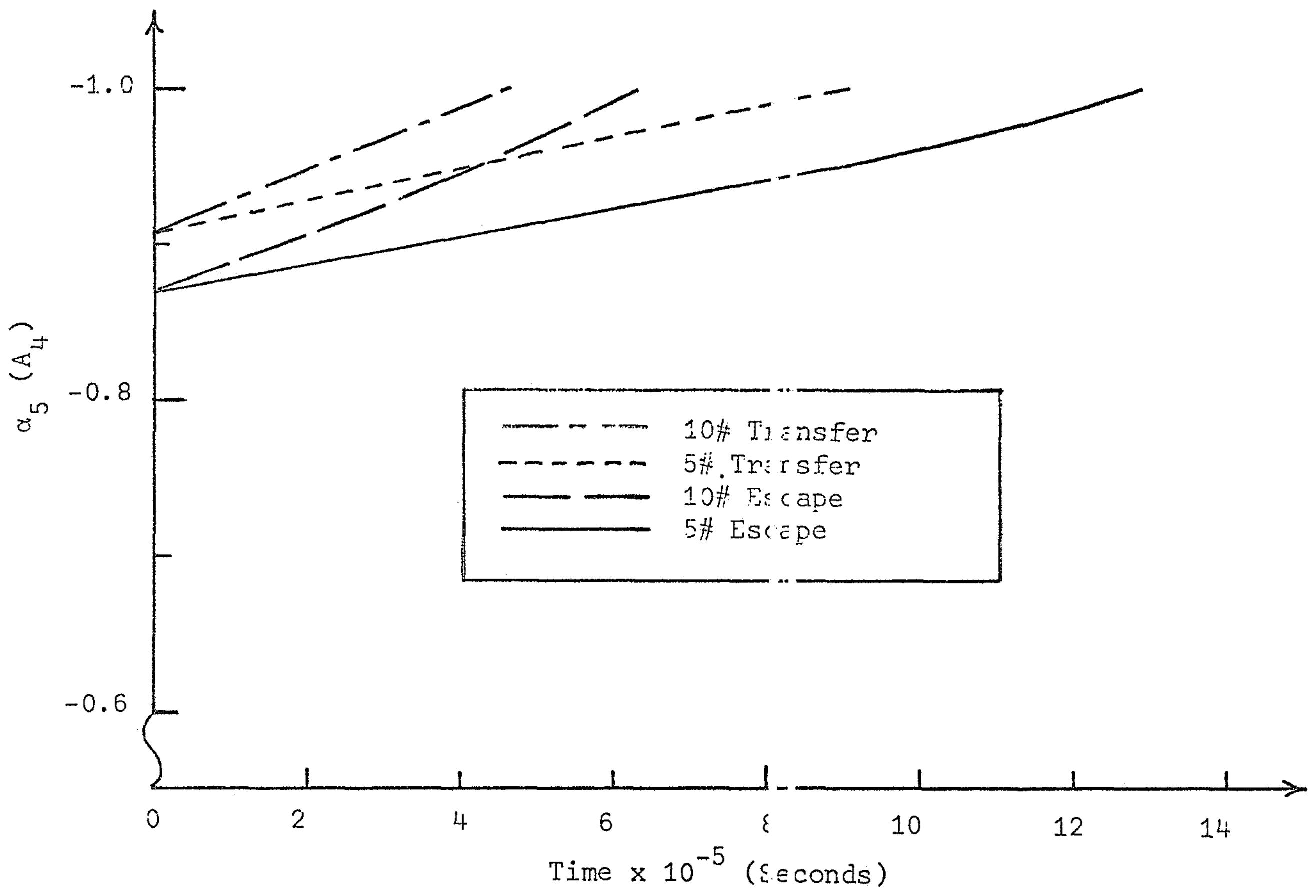


FIGURE 7. Time Histories of the Canonic Variables Generated by Λ_5

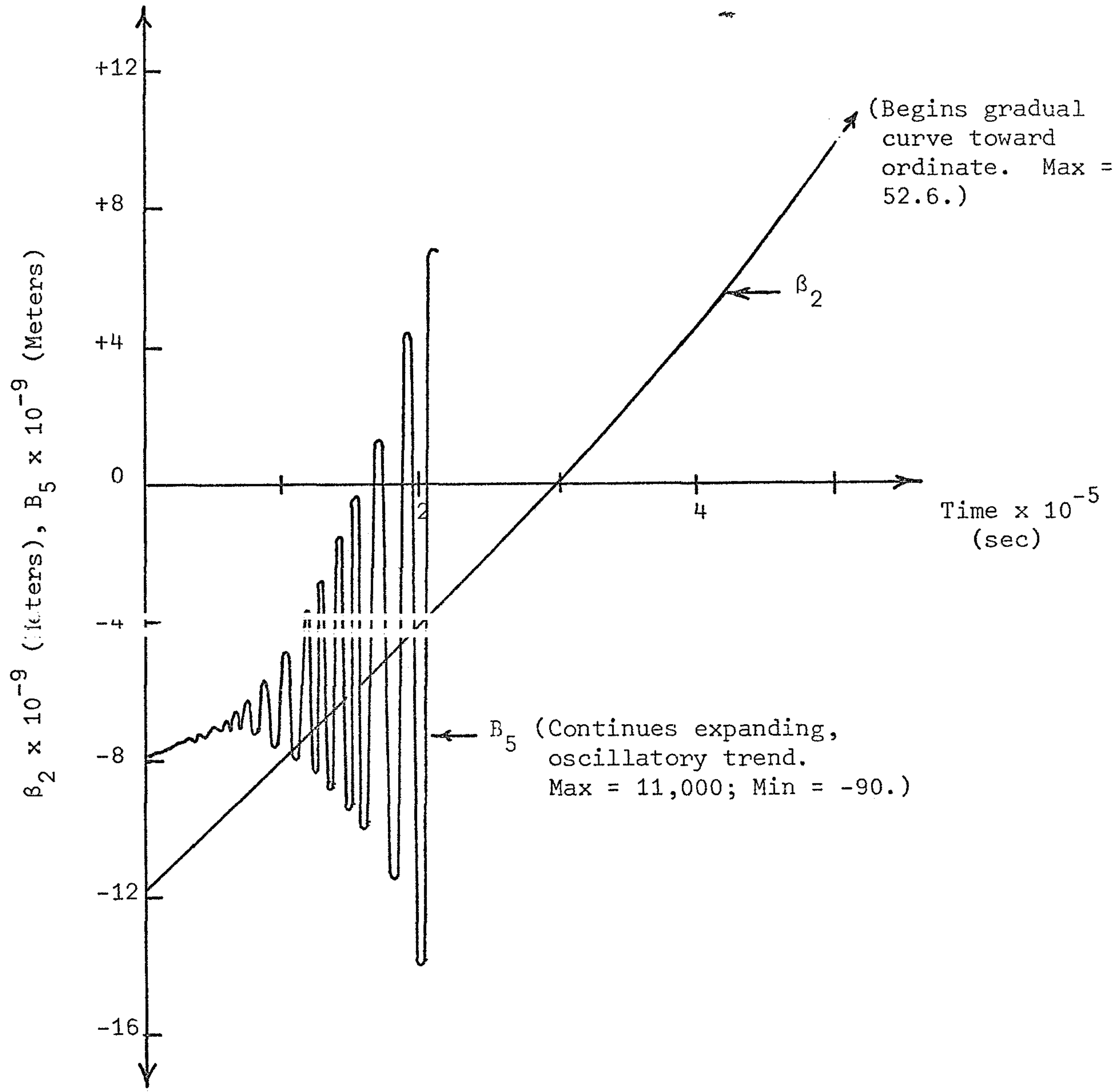


FIGURE 8. Partial Time Histories of The Canonic Variables Generated by Λ_1 (5# Transfer)

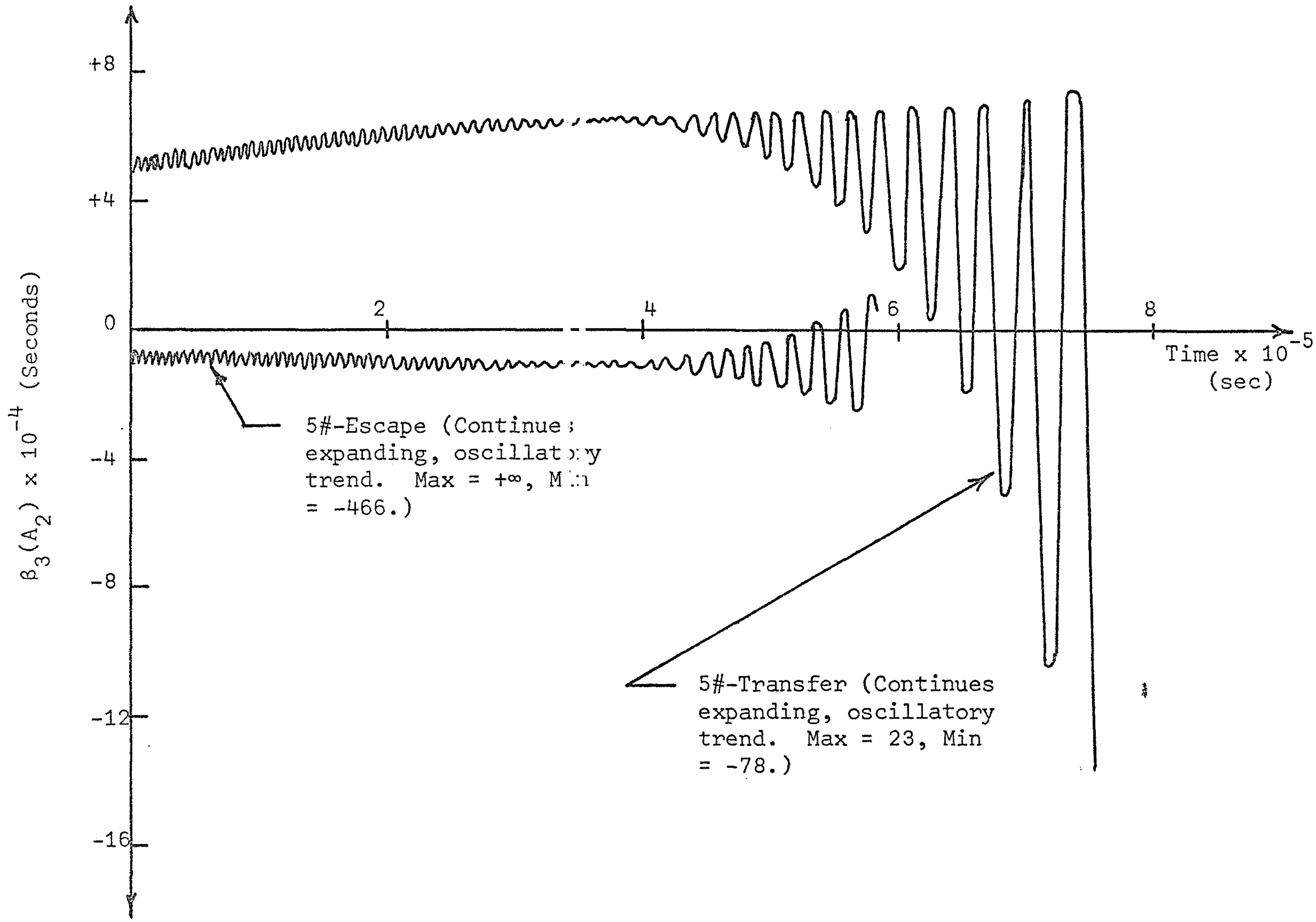


FIGURE 9. Partial Time Histories of The Canonic Variables Generated by Λ_2

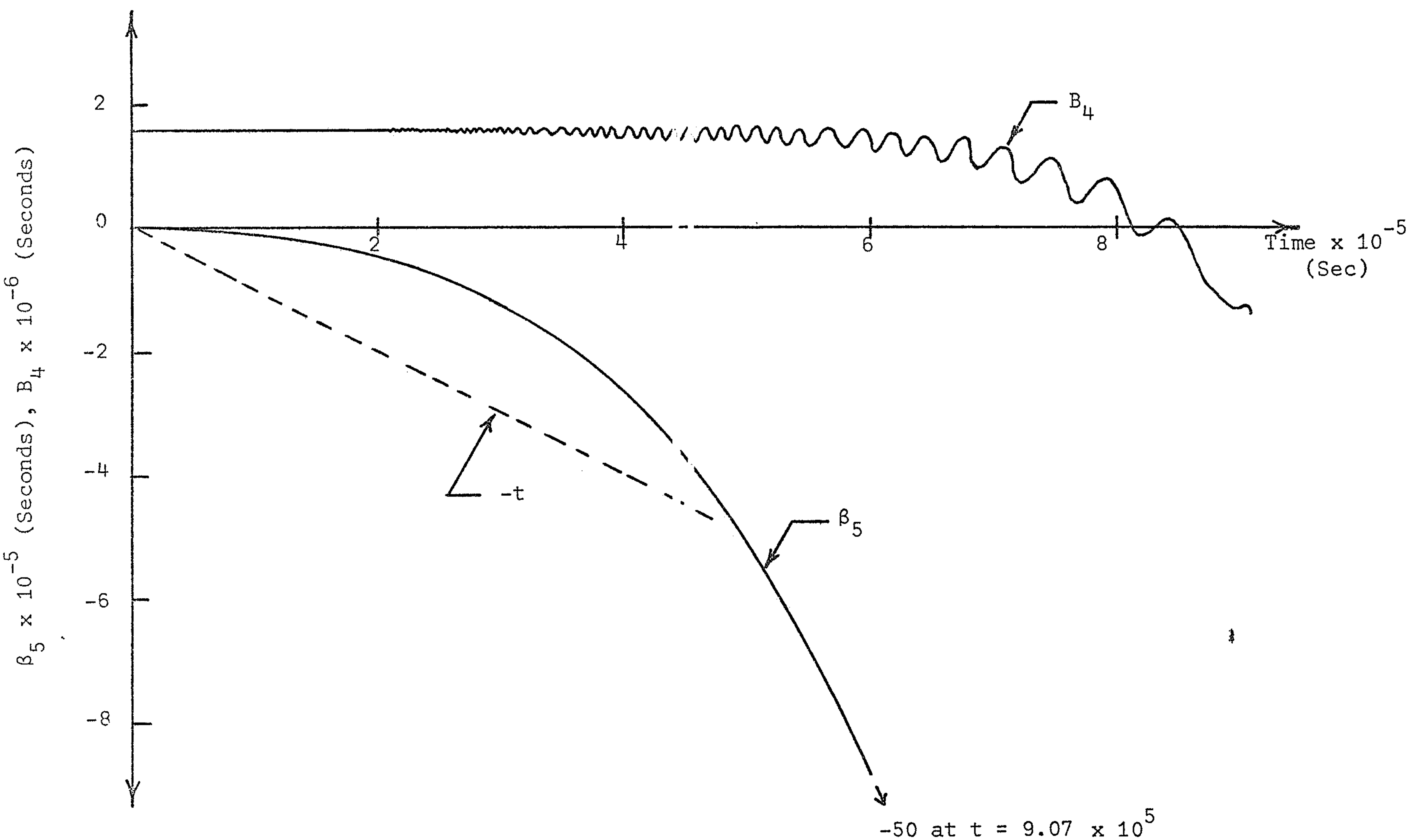


FIGURE 10. Time Histories of the Canonic Variables Generated by t (5#-Transfer)

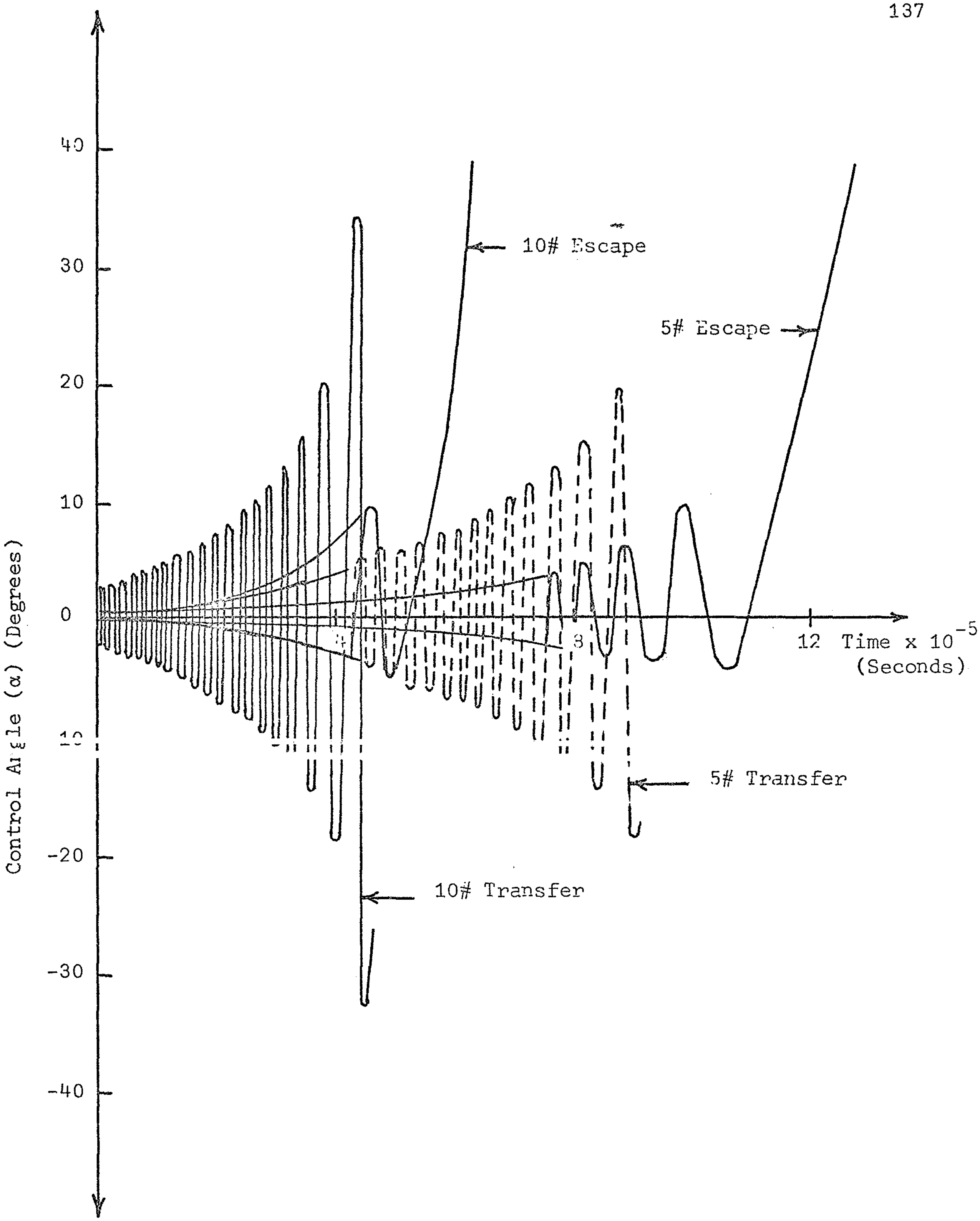


FIGURE 11. Optimal Control Angle for Four Representative Trajectories

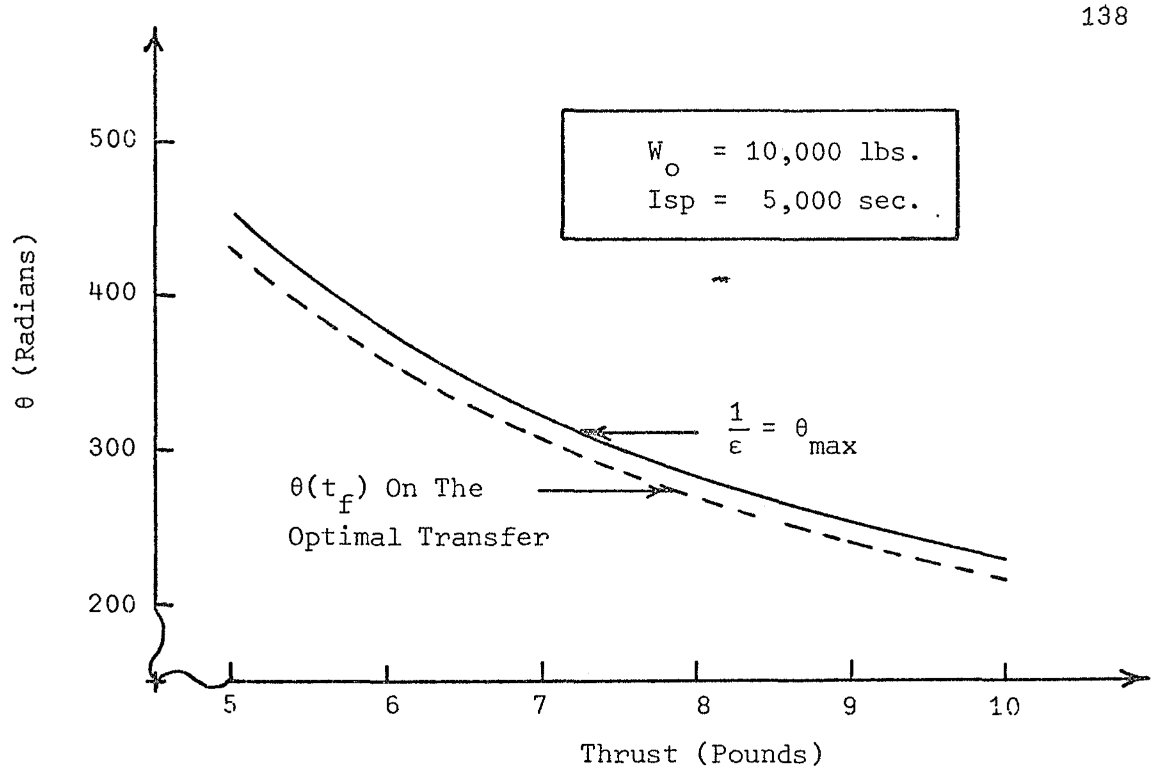


FIGURE 12a. Verification of The $1 > \varepsilon\theta$ Property On Typical Optimal Circular Transfers

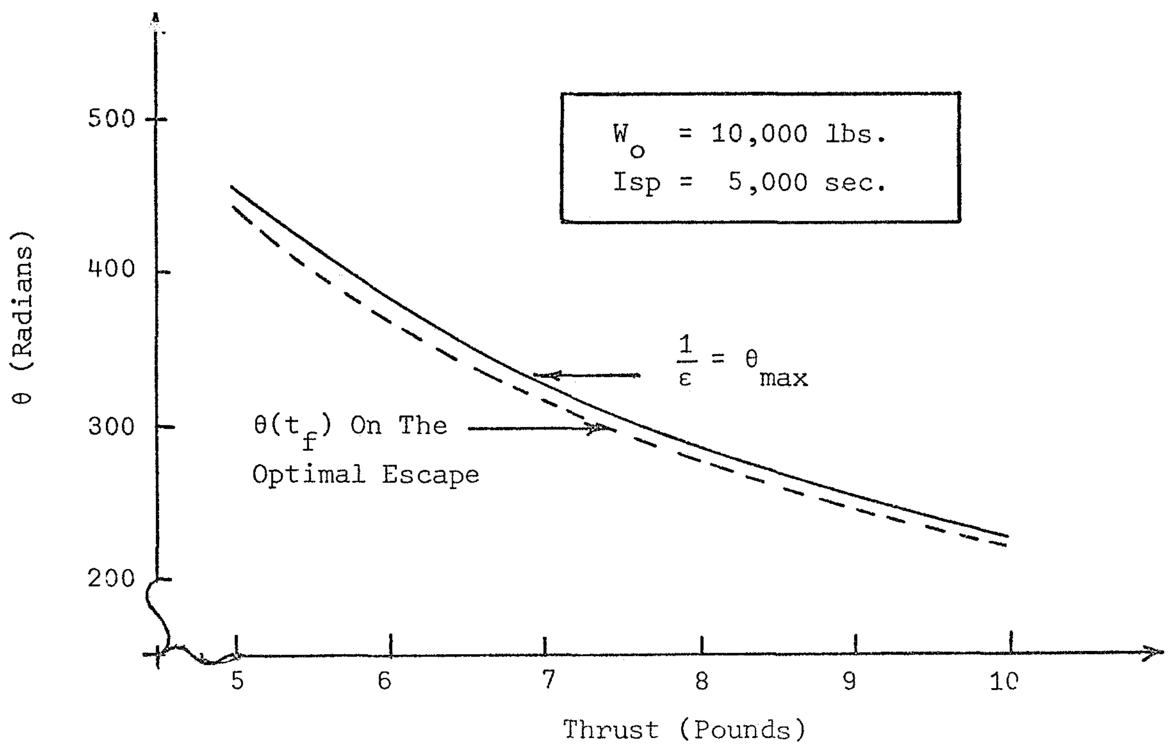


FIGURE 12b. Verification of The $1 > \varepsilon\theta$ Property On Typical Optimal Escape Trajectories

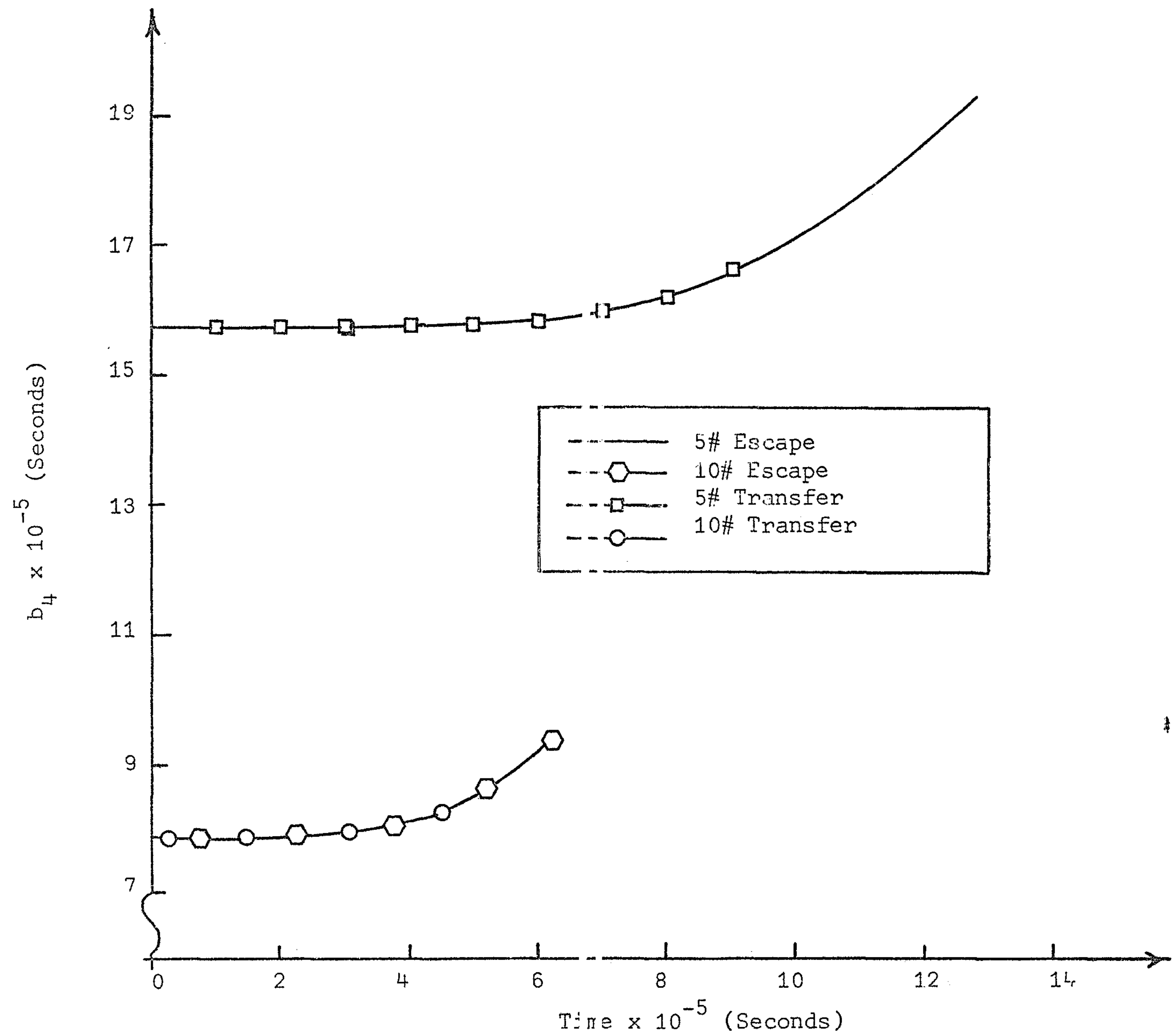


FIGURE 13. The Evaluation of b_4 (of Section 4.1) Along Four Representative Optimal Trajectories

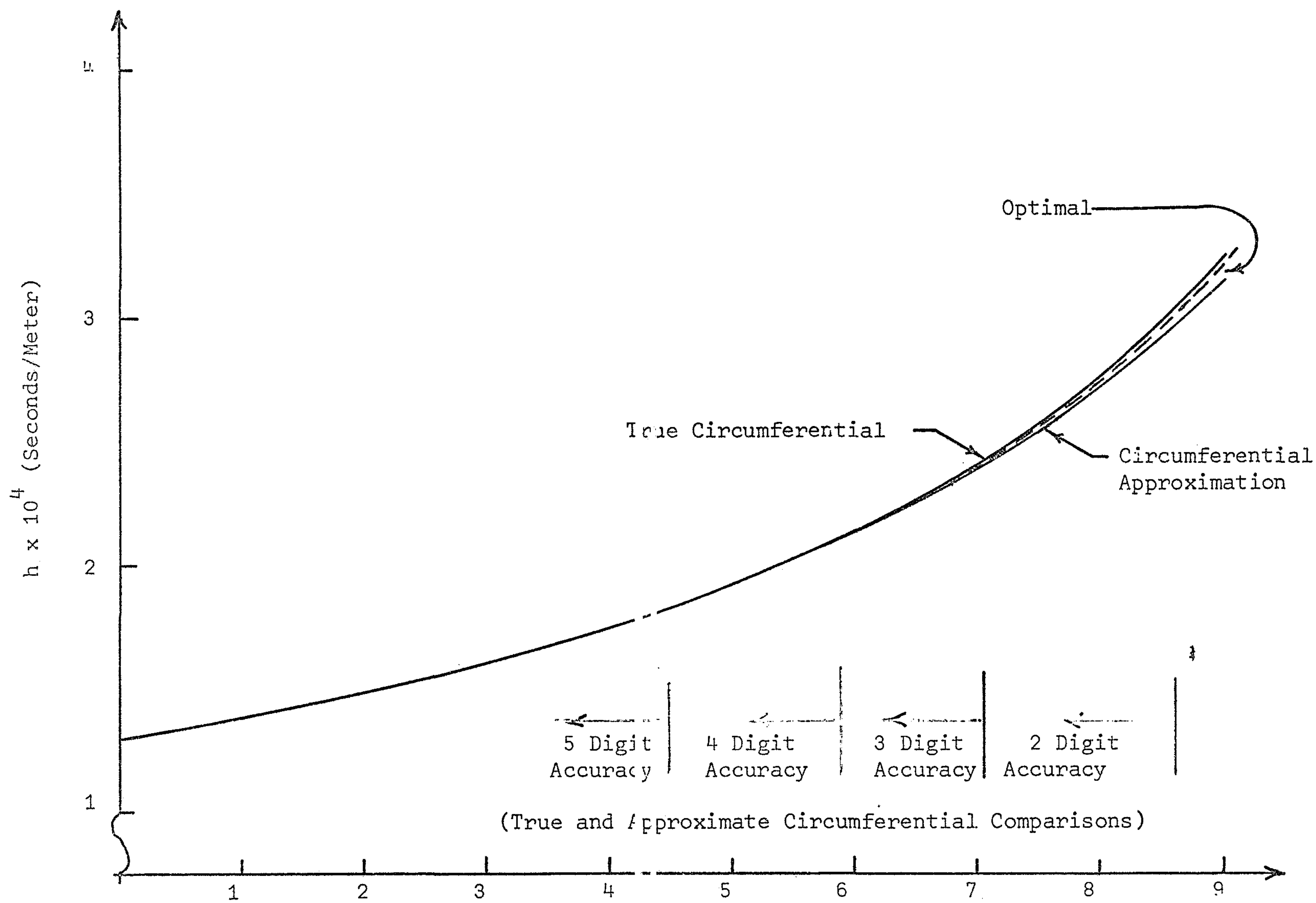


FIGURE 14. Time Histories of h for The 5#-Circular Orbit Transfer.

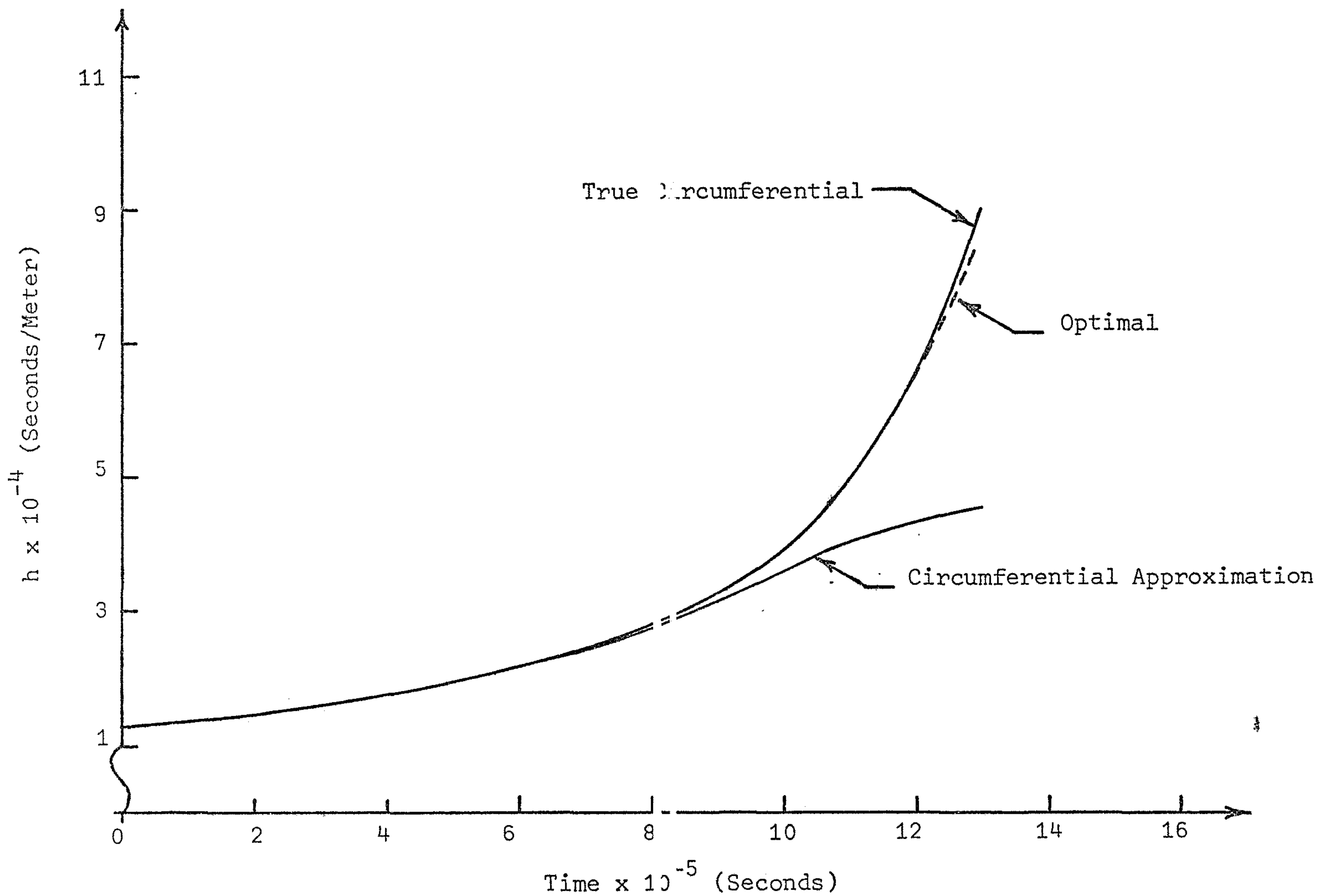


FIGURE 15. Time Histories of h for The 5#-Escape Trajectory

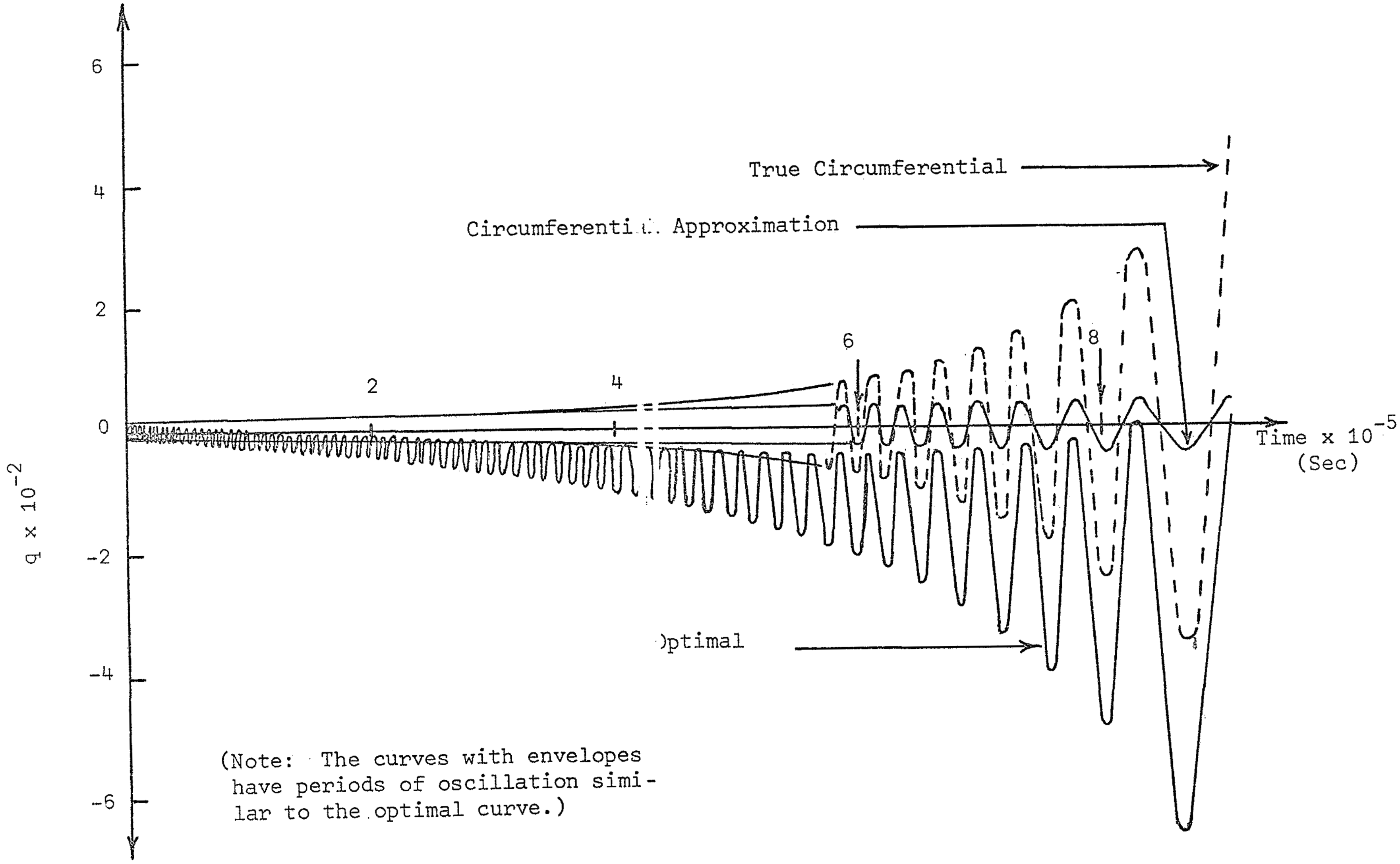


FIGURE 16. Time Histories of q for The 5#-Circular Orbit Transfer

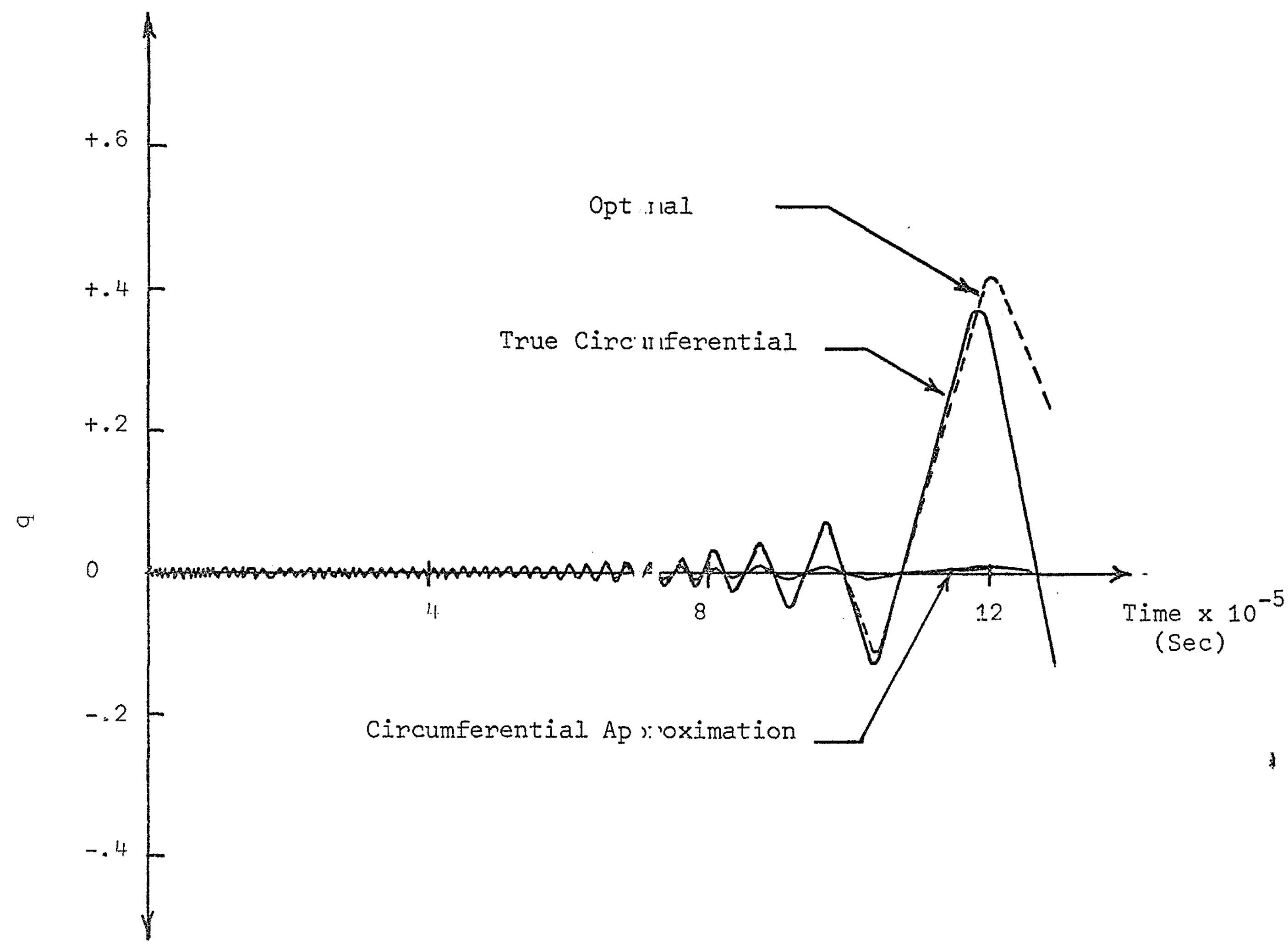


FIGURE 17. Time Histories of for The 5#-Escape Trajectory

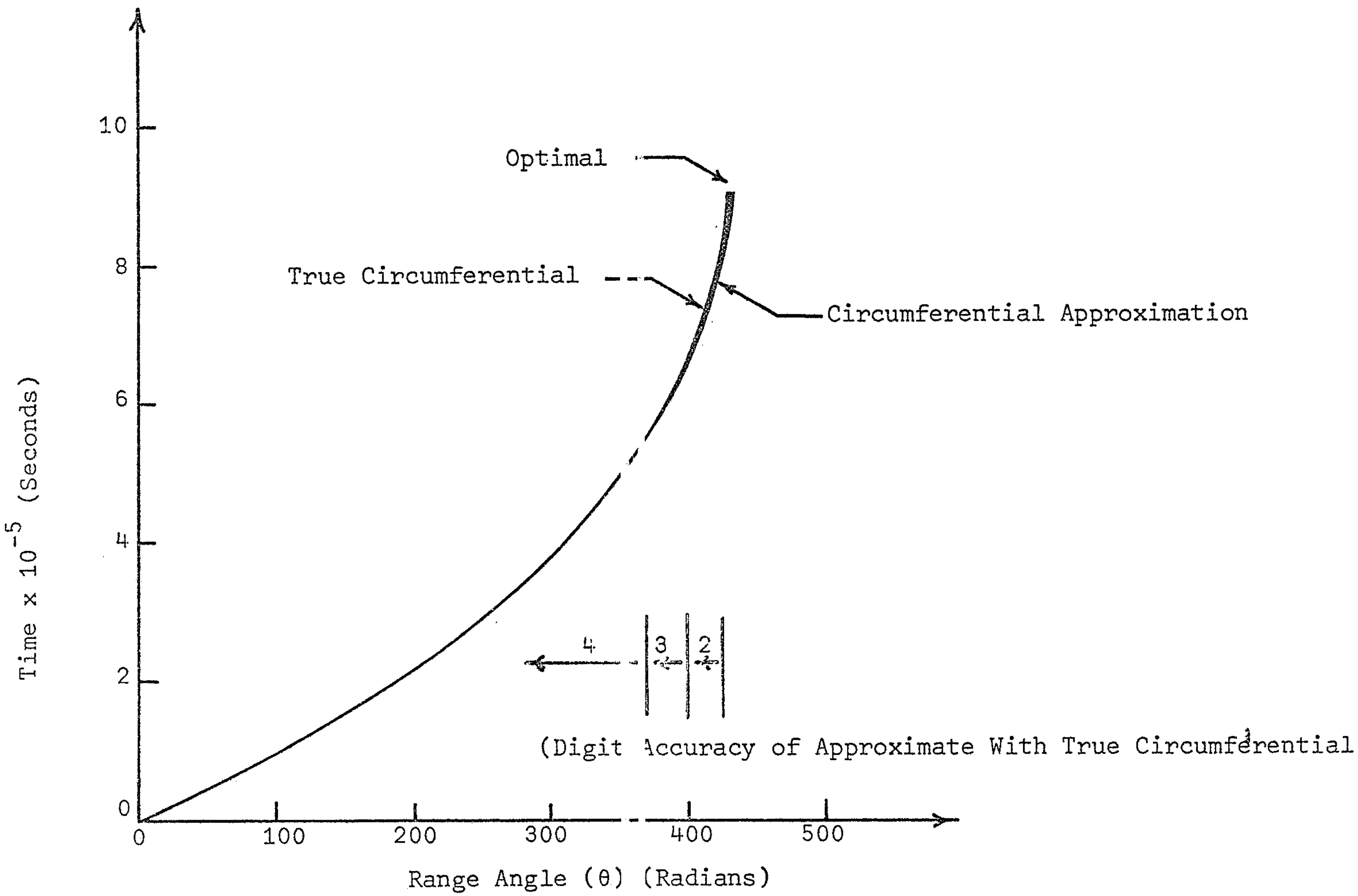


FIGURE 18. Time vs. Range Angle for The 5#-Circular Orbit Transfer

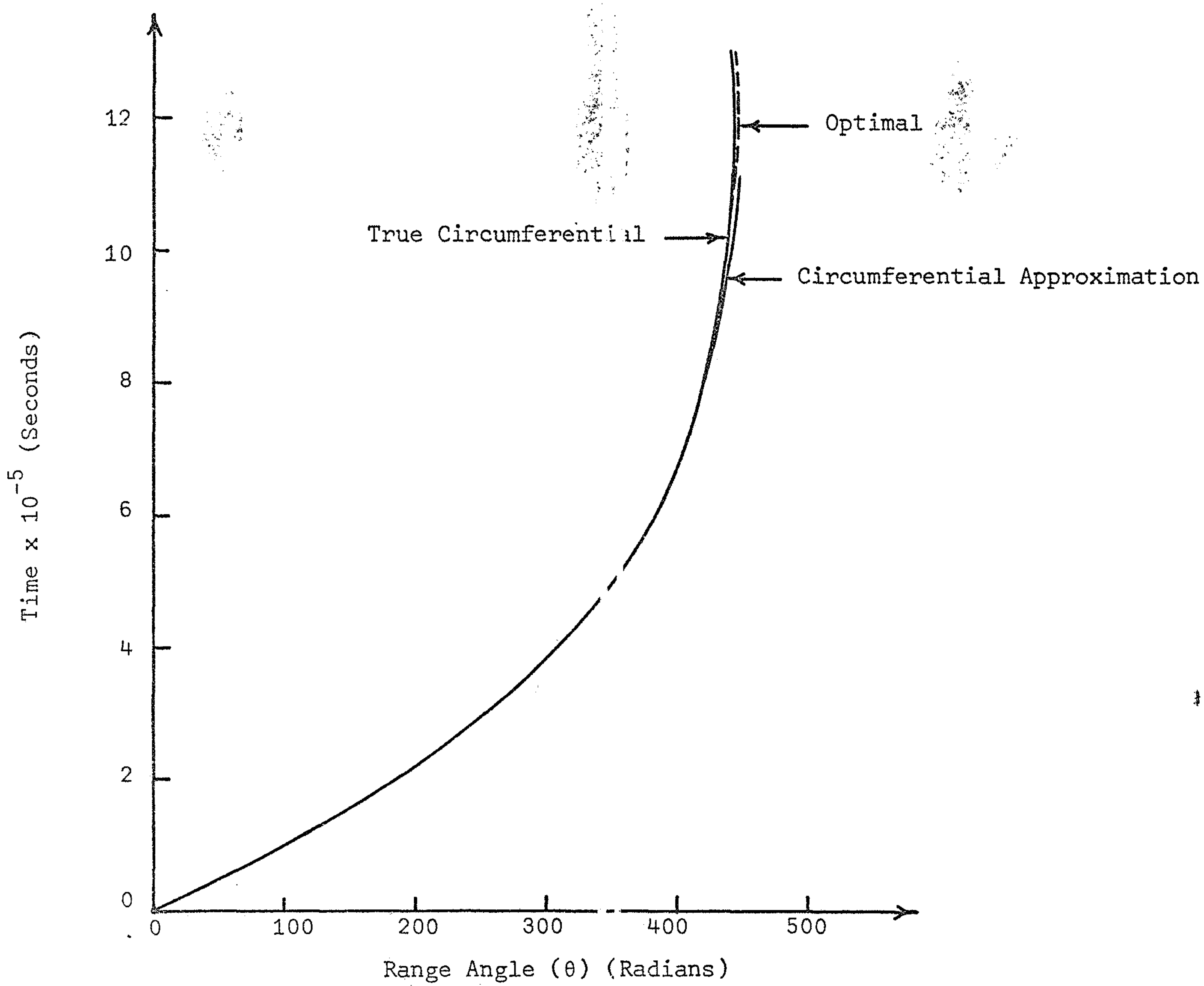


FIGURE 19. Time vs. Range Angle for the 5#-Escape Trajectory

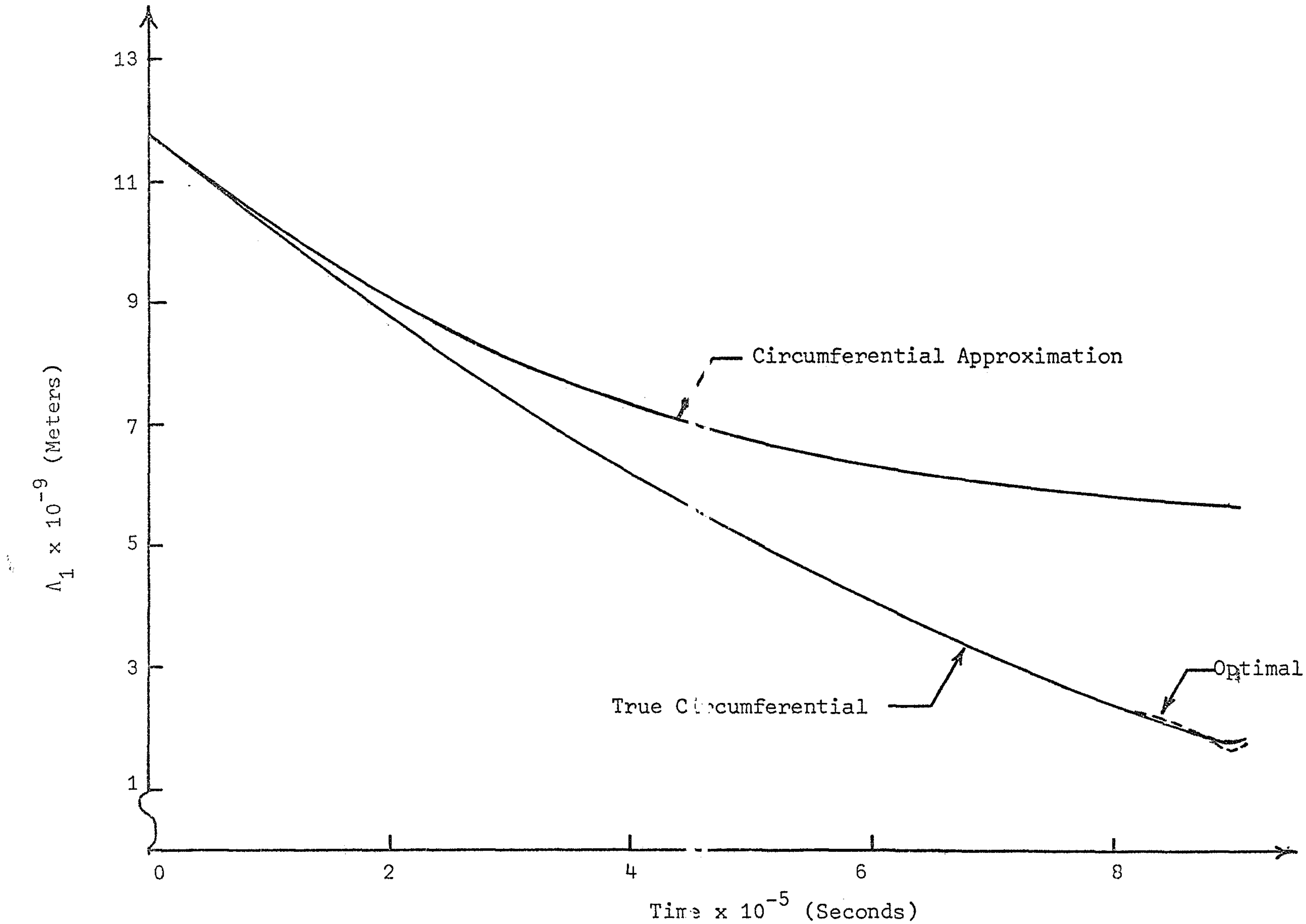


FIGURE 20. Time Histories of Λ_1 for The 5#-Circular Orbit Transfer

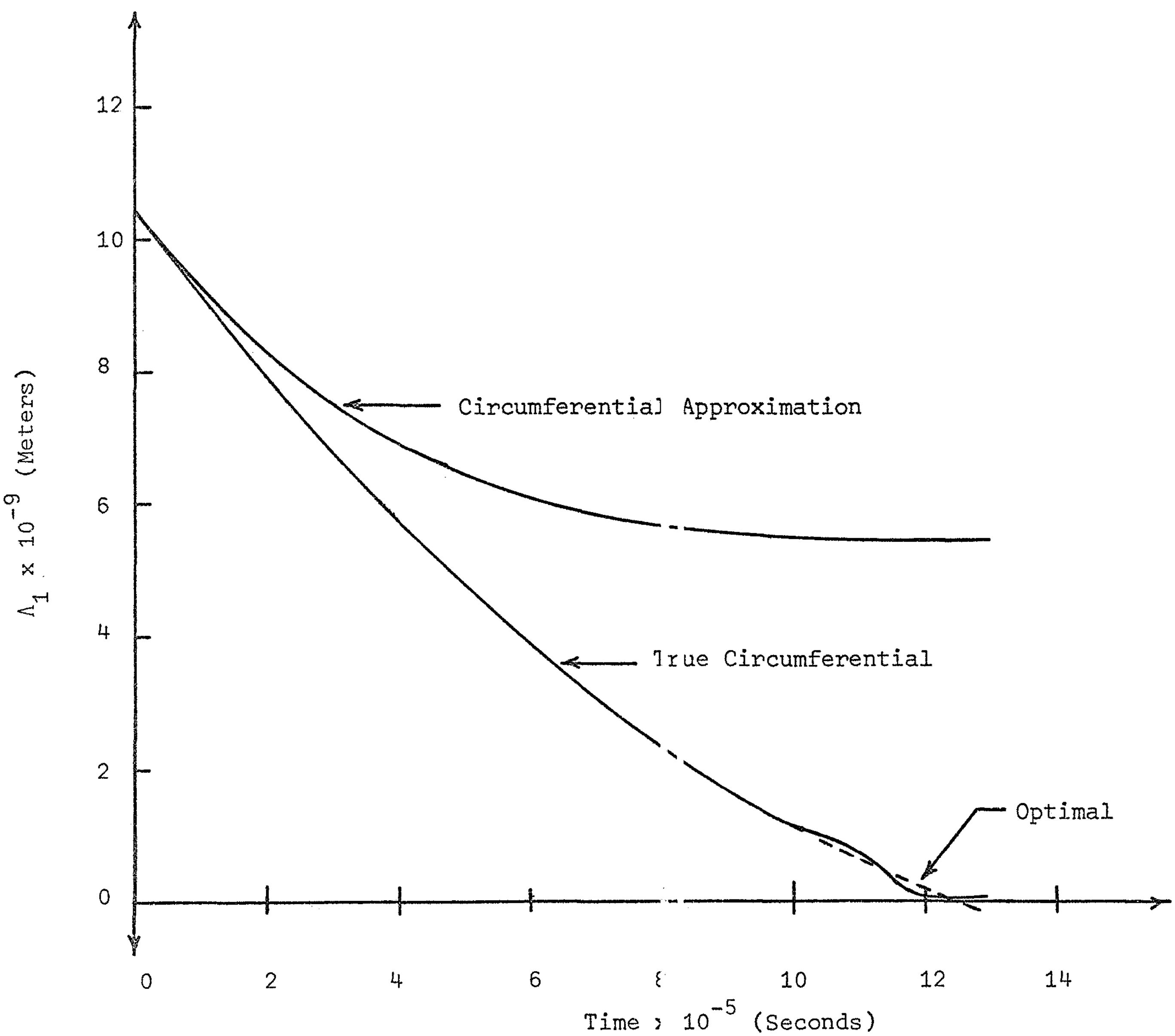


FIGURE 21. Time Histories of Λ_1 for The 5#-Escape Trajectory

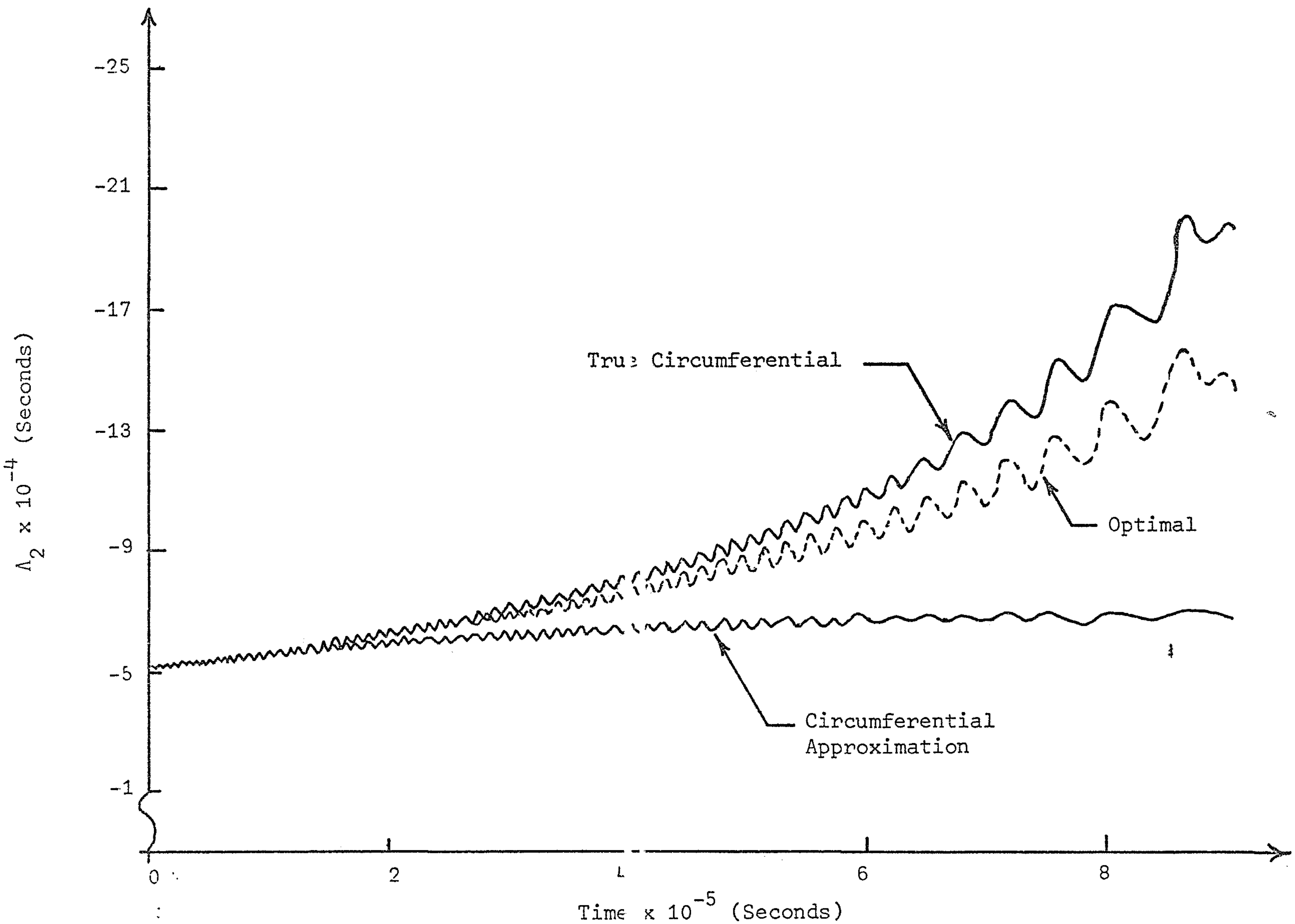


FIGURE 22. Time Histories of Λ_2 for The 5#-Circular Orbit Transfer

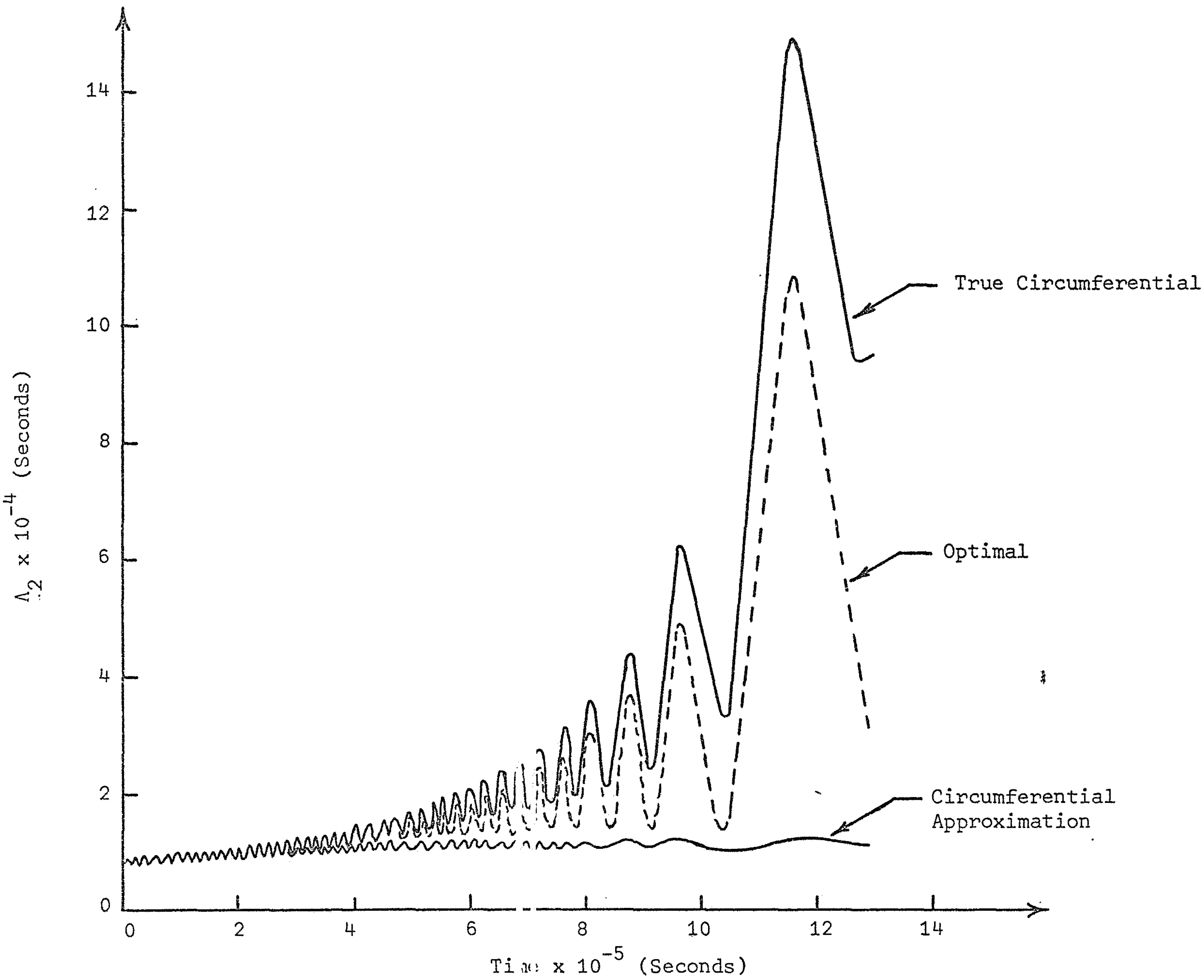


FIGURE 23. Time Histories of Λ_2 for The 5#-Escape Trajectory

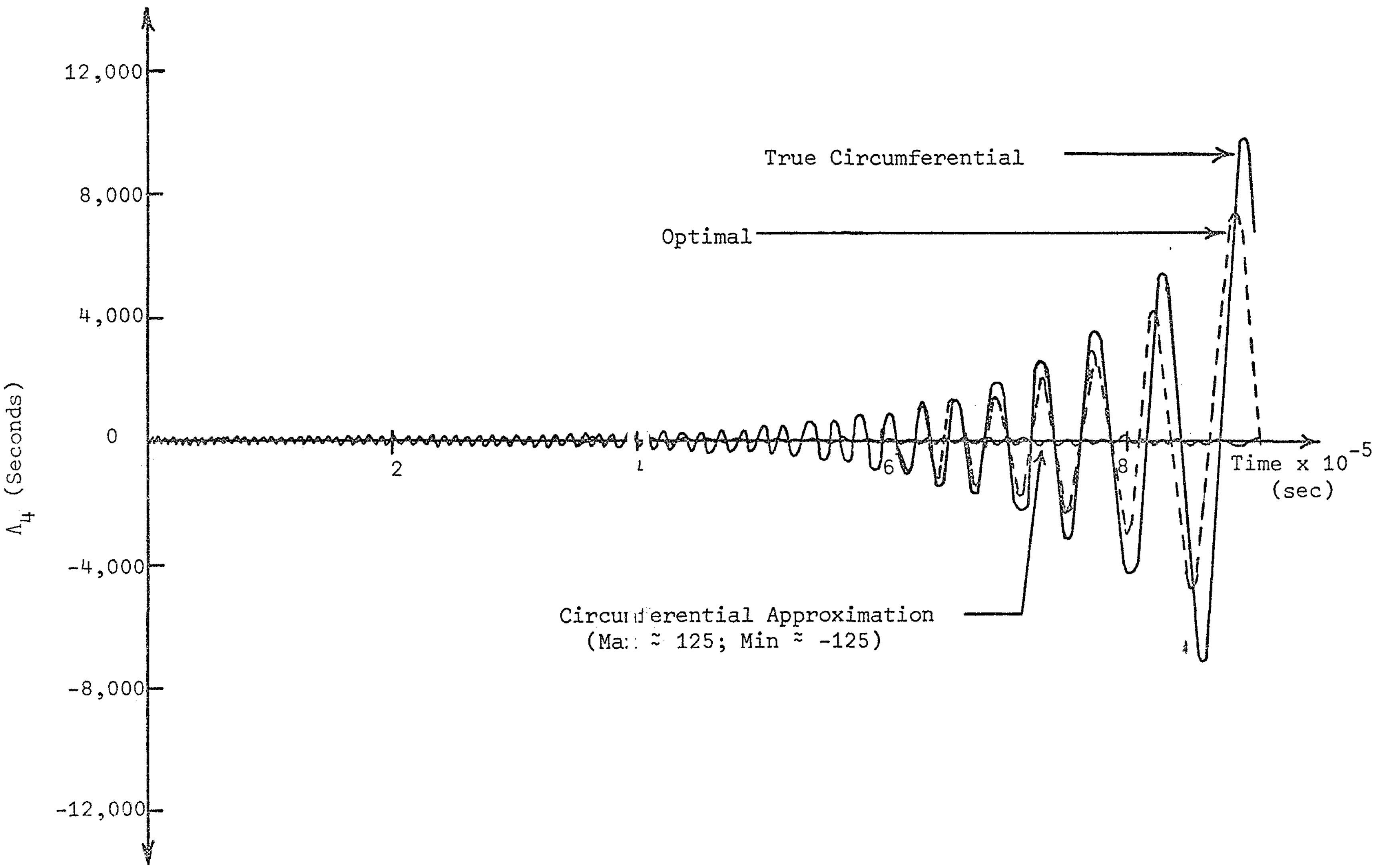


FIGURE 24. Time Histories of Λ_4 for The 5#-Circular Orbit Transfer

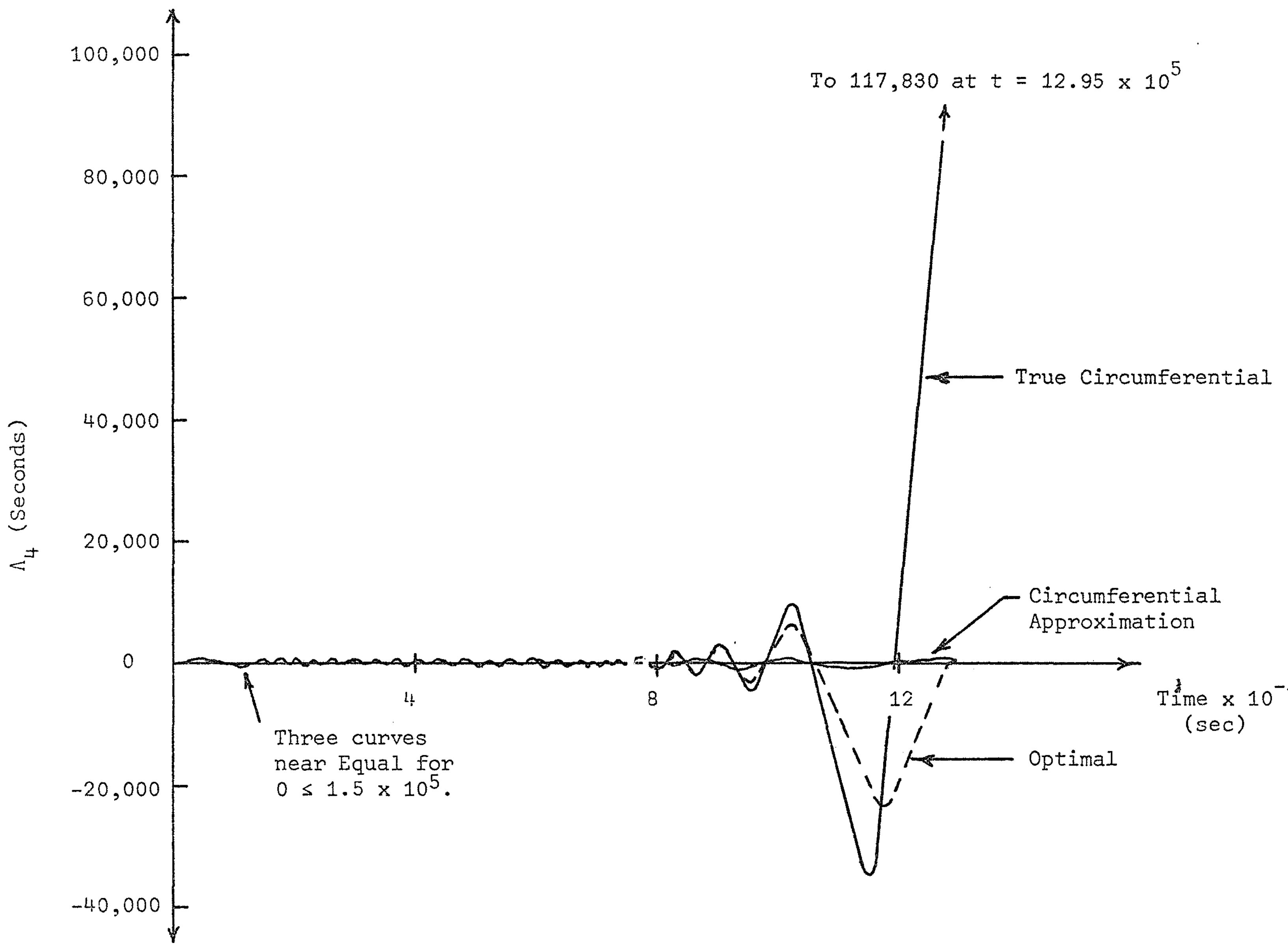


FIGURE 25. Time Histories of Λ_4 for The 5#-Escape Trajectory

TABLE 1

OPTIMAL ESCAPE TRAJECTORIES

W_o = 10,000 lbs. , Isp = 5,000 sec. , λ_4 = 0 .

Thrust (lbs)	$\lambda_1(t_o)$	$\lambda_2(t_o)$	$\lambda_3(t_o)$	t_f(sec)
10	-2.6033×10^{-3}	1.0000	1.1385×10^{-3}	623,843
9	$+6.4165 \times 10^{-3}$	1.0000	1.1413×10^{-3}	696,206
8	$+1.2936 \times 10^{-2}$	1.0000	1.1514×10^{-3}	786,950
7	-6.8179×10^{-3}	1.0000	1.1432×10^{-3}	903,996
6	-7.0256×10^{-3}	1.0000	1.1705×10^{-3}	1,060,715
5	-9.7592×10^{-3}	1.0000	1.1498×10^{-3}	1,281,087

(NOTE: $\frac{v_o}{r_o} = 1.1569 \times 10^{-3} \approx \lambda_3(t_o)$.)

TABLE 2

OPTIMAL CIRCULAR ORBIT TRANSFERS

$W_o = 10{,}000$ lbs. , $\Lambda_4 = 0$.

Thrust (lbs.)	ISP (sec)	$\lambda_1(t_o)$	$\lambda_2(t_o)$	$\lambda_3(t_o)$	t_f(optimal) (sec.)	t_f(approx.) (sec.)
10	8,000	2.6305×10^{-2}	1.0000	1.0894×10^{-3}	466,062	461,440
10	7,000	5.1479×10^{-2}	1.0000	1.1151×10^{-3}	464,216	460,402
10	6,000	5.9816×10^{-2}	1.0000	1.1667×10^{-3}	461,730	458,794
10	5,000	1.7365×10^{-2}	1.0000	1.2207×10^{-3}	458,197	456,690
9	5,000	-5.3134×10^{-2}	1.0000	1.1829×10^{-3}	507,861	507,433
8	5,000	-2.0539×10^{-2}	1.0000	1.0930×10^{-3}	570,179	570,862
7	5,000	-3.8289×10^{-2}	1.0000	1.1125×10^{-3}	650,275	652,414
6	5,000	-4.6497×10^{-2}	1.0000	1.1463×10^{-3}	757, 196	761,149
5	5,000	2.0779×10^{-2}	1.0000	1.1924×10^{-3}	907,167	913,379

(NOTE: $\frac{v_o}{r_o} = 1.1512 \times 10^{-3} \approx \lambda_3(t_o)$.)

APPENDIX A

In this appendix, both Theorem (2.1) and Property (2.1) of Chapter 2 will be proved. The proofs are expanded versions of the arguments given by Wintner[47] on pages 23-25 (for Theorem (2.1)) and page 29 (for Property (2.1)).

<u>Theorem 2.1</u>: Let $\{x(q, p, t), \lambda(q, p, t)\} \in C^2$ be a transformation which satisfies the conditions of the implicit function theorem, and let M be the Jacobian matrix of the transformation. Then, $\{x(q, p, t), \lambda(q, p, t)\}$ is a canonical transformation if and only if M is a symplectic matrix.

<u>Proof</u>: (<u>Sufficiency</u> $\leftarrow$) Consider the time derivatives of the set $\{x, \lambda\}$

$$\dot{x}_i = \sum_{j=1}^{n} \left[\frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial p_j} \dot{p}_j \right] + \frac{\partial x_i}{\partial t}$$

$$(i = 1, \ldots, n)$$

$$\dot{\lambda}_i = \sum_{j=1}^{n} \left[\frac{\partial \lambda_i}{\partial q_j} \dot{q}_j + \frac{\partial \lambda_i}{\partial p_j} \dot{p}_j \right] + \frac{\partial \lambda_i}{\partial t} .$$

Let M be the Jacobian matrix of the transformation, i.e.,

$$M \equiv \left[\begin{array}{c|c} \dfrac{\partial x_i}{\partial q_j} & \dfrac{\partial x_i}{\partial p_j} \\ \hline \dfrac{\partial \lambda_i}{\partial q_j} & \dfrac{\partial \lambda_i}{\partial p_j} \end{array} \right]$$

Then, in matrix form

$$\begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} = M \begin{bmatrix} \dot{q} \\ \\ \dot{p} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial t} \\ \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} . \tag{A.1}$$

Since the given transformation satisfies the implicit function theorem, the inverse transformation exists, i.e., $\{q(x, \lambda, t), p(x, \lambda, t)\}$. Thus,

$$\begin{bmatrix} \dot{q} \\ \\ \dot{p} \end{bmatrix} = N \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} , \tag{A.2}$$

where N is the Jacobian matrix for the inverse transformation, i.e.,

$$N \equiv \left[\begin{array}{c|c} \frac{\partial q_i}{\partial x_j} & \frac{\partial q_i}{\partial \lambda_j} \\ \hline \frac{\partial p_i}{\partial x_j} & \frac{\partial p_i}{\partial \lambda_j} \end{array}\right]$$

It will be shown now that $N = M^{-1}$. Substitution of Equation (A.2) into Equation (A.1) gives

$$\begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} = MN \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} + M \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial t} \\ \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} .$$

This equation must hold for <u>all</u> Hamiltonian functions, in particular those which are identically zero. Thus, $\dot{x} = \dot{\lambda} = 0$ so

$$M \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial x}{\partial t} \\ \\ \frac{\partial \lambda}{\partial t} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} = -M^{-1} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \\ \frac{\partial \lambda}{\partial t} \end{bmatrix}$$

and, then

$$\begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} = MN \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} \rightarrow MN = I .$$

Since M^{-1} exists (M is symplectic) and since $MN = I$, it follows that $N = M^{-1}$. By Property (5.1) of Chapter 2, $N = M^{-1} = -\frac{1}{\mu} JM^TJ$. Hence, Equation (A.2) can be expressed as

$$\begin{bmatrix} \dot{q} \\ \\ \dot{p} \end{bmatrix} = -\frac{1}{\mu} JM^TJ \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix}.$$

Upon multiplication by J

$$J \begin{bmatrix} \dot{q} - \frac{\partial q}{\partial t} \\ \\ \dot{p} - \frac{\partial p}{\partial t} \end{bmatrix} = -\frac{1}{\mu} J^2M^TJ \begin{bmatrix} \dot{x} \\ \\ \dot{\lambda} \end{bmatrix}.$$

But, $J \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \dot{\lambda} \\ -\dot{x} \end{bmatrix}$ and $J^2 = -I$ so

$$\begin{bmatrix} \dot{p} - \frac{\partial p}{\partial t} \\ \\ -(\dot{q} - \frac{\partial q}{\partial t}) \end{bmatrix} = \frac{1}{\mu} M^T \begin{bmatrix} \dot{\lambda} \\ \\ -\dot{x} \end{bmatrix}.$$

Since $\{H, x, \lambda\}$ is a Hamiltonian system

$$\begin{bmatrix} \dot{p} - \frac{\partial p}{\partial t} \\ \\ -(\dot{q} - \frac{\partial q}{\partial t}) \end{bmatrix} = -\frac{1}{\mu} M^T \begin{bmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial \lambda} \end{bmatrix}. \qquad (A.3)$$

In scalar form Equation (A.3) becomes

$$\dot{p}_i - \frac{\partial p_i}{\partial t} = -\frac{1}{\mu}\sum_{j=1}^{n}\left[\frac{\partial x_j}{\partial q_i}\frac{\partial H}{\partial x_j} + \frac{\partial \lambda_j}{\partial q_i}\frac{\partial H}{\partial \lambda_j}\right]$$

$$-(\dot{q}_i - \frac{\partial q_i}{\partial t}) = -\frac{1}{\mu}\sum_{j=1}^{n}\left[\frac{\partial x_j}{\partial p_i}\frac{\partial H}{\partial x_j} + \frac{\partial \lambda_j}{\partial p_i}\frac{\partial H}{\partial \lambda_j}\right] . \tag{A.4}$$

Define the function

$$K(q, p, t) \equiv \frac{1}{\mu} H[x(q, p, t), \lambda(q, p, t), t] + R(q, p, t), \tag{A.5}$$

where $\frac{1}{\mu}$ and R are called the multiplier and the remainder function of the canonical transformation, respectively. Then,

$$\frac{\partial K}{\partial q_i} = \frac{1}{\mu}\sum_{j=1}^{n}\left[\frac{\partial H}{\partial x_j}\frac{\partial x_j}{\partial q_i} + \frac{\partial H}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial q_i}\right] + \frac{\partial R}{\partial q_i}$$

$$\frac{\partial K}{\partial p_i} = \frac{1}{\mu}\sum_{j=1}^{n}\left[\frac{\partial H}{\partial x_j}\frac{\partial x_j}{\partial p_i} + \frac{\partial H}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial p_i}\right] + \frac{\partial R}{\partial p_i} . \tag{A.6}$$

Substitution of Equations (A.6) into Equations (A.4) gives

$$-\dot{p}_i + \frac{\partial p_i}{\partial t} = \frac{\partial K}{\partial q_i} - \frac{\partial R}{\partial q_i}$$

$$\dot{q}_i - \frac{\partial q_i}{\partial t} = \frac{\partial K}{\partial p_i} - \frac{\partial R}{\partial p_i} . \tag{A.7}$$

In the definition of K, given by Equation (A.5), the term $\frac{1}{\mu} H$ is well-defined, but R is not. Thus, for K to be a Hamiltonian function in the $\{q, p\}$-space, it must be shown that there exists a function $R(q, p, t)$ which satisfies the equations

$$-\frac{\partial p_i}{\partial t} = \frac{\partial R}{\partial q_i} \qquad \frac{\partial q_i}{\partial t} = \frac{\partial R}{\partial p_i} \;. \tag{A.8}$$

For then Equations (A.7) become

$$\begin{aligned} \dot{p}_i &= -\frac{\partial K}{\partial q_i} \\ \dot{q}_i &= \frac{\partial K}{\partial p_i} \,, \end{aligned} \qquad (i = 1,\ldots,n)$$

which is a Hamiltonian system in the {q, p}-space.

Thus, to complete the sufficiency proof, it must be shown that there exists a solution $R(q, p, t)$ to Equations (A.8) (note that the solution need not be unique). In matrix form, Equations (A.8) can be written as

$$\begin{bmatrix} \frac{\partial R}{\partial p} \\ \\ \frac{\partial R}{\partial q} \end{bmatrix} = \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ -\frac{\partial p}{\partial t} \end{bmatrix} = J \begin{bmatrix} \frac{\partial p}{\partial t} \\ \\ \frac{\partial q}{\partial t} \end{bmatrix} .$$

or

$$-\begin{bmatrix} \frac{\partial R}{\partial q} \\ \\ \frac{\partial R}{\partial p} \end{bmatrix} = J \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix} .$$

Note that the left-hand side of this equation is a gradient. Thus, a solution of Equations (A.8) exists if the vector

$$J \begin{bmatrix} \frac{\partial q}{\partial t} \\ \\ \frac{\partial p}{\partial t} \end{bmatrix}$$

is a gradient. Since $\{x(q, p, t), \lambda(q, p, t)\}$ is assumed to be of class C^2, then the functions $\{ \frac{\partial q_1}{\partial t}, \ldots, \frac{\partial q_n}{\partial t}, \frac{\partial p_1}{\partial t}, \ldots, \frac{\partial p_n}{\partial t} \}$ are of class C^1. Thus, if there exists a function $R(q, p, t)$, then $\{ \frac{\partial R}{\partial q_1}, \ldots, \frac{\partial R}{\partial q_n}, \frac{\partial R}{\partial p_1}, \ldots, \frac{\partial R}{\partial p_n} \}$ must be of class C^1 and therefore the following relations will be true

$$\frac{\partial^2 R}{\partial q_i \partial q_j} = \frac{\partial^2 R}{\partial q_j \partial q_i}, \quad \frac{\partial^2 R}{\partial q_i \partial p_j} = \frac{\partial^2 R}{\partial p_j \partial q_i}, \quad \frac{\partial^2 R}{\partial p_i \partial p_j} = \frac{\partial^2 R}{\partial p_j \partial p_i} . \tag{A.9}$$

To ease the notation, let the two sets of variables be denoted by

$$\begin{aligned} \{X_1, \ldots, X_{2n}\} &\equiv \{x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n\} \\ \{Q_1, \ldots, Q_{2n}\} &\equiv \{q_1, \ldots, q_n, p_1, \ldots, p_n\} , \end{aligned} \tag{A.10}$$

and let

$$\begin{aligned} X &= \phi(Q, t) \\ Q &= \psi(X, t) \end{aligned}$$

represent the given transformation and its inverse, respectively. Then, in summary: if $[\frac{\partial R}{\partial Q}]$ is a gradient of class C^1, Equations (A.9) must be satisfied. Thus, there exists a solution $R(Q)$ of Equations (A.8) only if $J[\frac{\partial \psi}{\partial t}]$ is a gradient of class C^1, which implies that $\frac{\partial}{\partial Q} \{J[\frac{\partial \psi}{\partial t}]\}$ is a symmetric matrix. It will be shown that this is indeed the case.

By expanding each side of the following equality it is readily determined that

$$\frac{\partial}{\partial Q} \{J[\frac{\partial \psi}{\partial t}]\} = J\{\frac{\partial}{\partial Q} [\frac{\partial \psi}{\partial t}]\} \tag{A.11}$$

Another convenient representation is given by the following lemma.

Lemma 1: $\frac{\partial}{\partial Q}\left[\frac{\partial \psi}{\partial t}\right] = \frac{\partial N}{\partial t} N^{-1}$, where N is the Jacobian of the inverse transformation.

Proof: Consider the inverse transformation $Q = \psi(X, t)$. Then,

$$\frac{\partial \psi}{\partial X} = \frac{\partial \psi[\phi(Q, t), t]}{\partial X} \quad \text{and} \quad \frac{\partial \psi}{\partial t} = \frac{\partial \psi[\phi(Q, t), t]}{\partial t}$$

i.e., after the differentiations are performed the relation $X = \phi(Q, t)$ is used to form a function of $\{Q, t\}$ again. Since $\psi(X, t) \varepsilon C^2$, it follows by the chain rule that

$$\frac{\partial}{\partial t}\left[\frac{\partial \psi}{\partial X}\right] = \frac{\partial}{\partial X}\left[\frac{\partial \psi}{\partial t}\right] = \frac{\partial}{\partial Q}\left[\frac{\partial \psi}{\partial t}\right]\frac{\partial \psi}{\partial X} \tag{A.12}$$

since $\frac{\partial \psi}{\partial t}$ is a function of $\{Q, t\}$. But, $N \equiv \frac{\partial \psi}{\partial X}$, so upon substitution in Equation (A.12), the following expression is obtained

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial Q}\left[\frac{\partial \psi}{\partial t}\right] N .$$

Then, since N^{-1} exists,

$$\frac{\partial}{\partial Q}\left[\frac{\partial \psi}{\partial t}\right] = \frac{\partial N}{\partial t} N^{-1} . \tag{A.13}$$

The representations of Equation (A.11) and Lemma 1 then give

$$\frac{\partial}{\partial Q}\{J[\frac{\partial \psi}{\partial t}]\} \equiv J \cdot \frac{\partial N}{\partial t} \cdot N^{-1} .$$

Thus, the problem is now to show that $J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}$ is symmetric, i.e.,

$$[\ J \cdot \frac{\partial N}{\partial t} \cdot N^{-1}\ ]^T \ = \ J \cdot \frac{\partial N}{\partial t} \cdot N^{-1} \ . \tag{A.14}$$

Since by hypothesis M is symplectic, it follows that N is symplectic since $N = M^{-1}$ and the symplectic matrices form a group. Thus, $N^TJN = \frac{1}{\mu} J$. Since $\frac{1}{\mu} J$ is just a matrix of constants, it follows that

$$\frac{\partial}{\partial t} [N^TJN] \ = \ \frac{\partial}{\partial t} [\ \frac{1}{\mu} J\ ] \ = \ 0$$

or,

$$(\ \frac{\partial N}{\partial t}\ )^T\ JN \ + \ N^TJ\ \frac{\partial N}{\partial t} \ = \ 0\ .$$

Since J is skew-symmetric, $J^T = -J$ so

$$-(\ \frac{\partial N}{\partial t}\ )^T\ J^TN \ + \ N^TJ\ \frac{\partial N}{\partial t} \ = \ 0\ .$$

Upon multiplication by N^{-1} first on the right and then $(N^T)^{-1}$ on the left leads to the following expression

$$(N^{-1})^T\ (\ \frac{\partial N}{\partial t}\ )^T\ J^T \ - \ J\ \frac{\partial N}{\partial t}\ N^{-1} \ = \ 0\ .$$

Therefore,

$$[J\ \frac{\partial N}{\partial t}\ N^{-1}]^T \ = \ J\ \frac{\partial N}{\partial t}\ N^{-1}\ ,$$

which verifies the symmetry of $J\ \frac{\partial N}{\partial t}\ N^{-1}$, and thus, there exists a solution $R(q, p, t)$ of Equations (A.8).

(<u>Necessity →</u>) Since $\{x(q, p, t), \lambda(q, p, t)\}$ is assumed to be canonical, there exists a $K(q, p, t)$ such that $\{\dot{q} = \frac{\partial K}{\partial p}\ , \dot{p} = -\ \frac{\partial K}{\partial q}\}$. It must be shown that M is symplectic (or, equivalently, that N is symplectic since $N = M^{-1}$ implies that M is symplectic if N is symplectic). Making use of the notation introduced in Equations (A.10), Equations (A.2) can be written as

$$\dot{Q} = M^{-1}\dot{X} + \frac{\partial Q}{\partial t}, \tag{A.15}$$

where

$$Q = \begin{bmatrix} q \\ p \end{bmatrix} \qquad X = \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$

But, $\{H, x, \lambda\} \equiv \{H, X\}$ is a Hamiltonian system so

$$\dot{X} \equiv \begin{bmatrix} \frac{\partial H}{\partial \lambda} \\ -\frac{\partial H}{\partial x} \end{bmatrix} = J \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial \lambda} \end{bmatrix} = J[\frac{\partial H}{\partial X}].$$

Substitution in Equation (A.15) gives

$$\dot{Q} = M^{-1} J[\frac{\partial H}{\partial X}] + \frac{\partial Q}{\partial t}. \tag{A.16}$$

But, $H(X, t) = H[X(Q, t), t]$ which implies (upon application of the chain rule) that

$$\frac{\partial H}{\partial Q} = [\frac{\partial X}{\partial Q}]^T \frac{\partial H}{\partial X} = M^T \frac{\partial H}{\partial X}.$$

Substitution in Equation (A.16) gives

$$\dot{Q} = M^{-1} J(M^T)^{-1} \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t}.$$

It is given that $\{K, q, p\} \equiv \{K, Q\}$ is a Hamiltonian system, so $\dot{Q} = J \frac{\partial K}{\partial Q}$ which implies

$$J \frac{\partial K}{\partial Q} = M^{-1} J(M^T)^{-1} \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t} . \tag{A.17}$$

Recalling that $J^{-1} = -J$ (Property C.3 of Chapter 2),

$$\frac{\partial K}{\partial Q} = - JM^{-1} J(M^T)^{-1} \frac{\partial H}{\partial Q} - J \frac{\partial Q}{\partial t} ,$$

or

$$\frac{\partial K}{\partial Q} = - JNJN^T \frac{\partial H}{\partial Q} - J \frac{\partial Q}{\partial t} ,$$

where use has been made of the matrix identity $(M^T)^{-1} = (M^{-1})^T$. Then,

$$\frac{\partial K}{\partial Q} = - J[NJN^T \frac{\partial H}{\partial Q} + \frac{\partial Q}{\partial t}] . \tag{A.18}$$

The left-hand side of Equation (A.18) is a gradient. It will now be shown that the right-hand side of Equation (A.18) is a gradient (for <u>every</u> Hamiltonian H) only if N is a symplectic matrix.

Since Equation (A.18) must hold for every H-function, in particular it must hold for $H \equiv 0$. Then,

$$\frac{\partial K}{\partial Q} = - J \frac{\partial Q}{\partial t}$$

which implies that $-J \frac{\partial Q}{\partial t}$ is a gradient with respect to Q . It then follows that $JNJN^T (\frac{\partial H}{\partial Q})$ must be a gradient. On observing this fact, the following Lemma can be stated.

Lemma 2: If $JNJN^T \frac{\partial H}{\partial Q}$ is a gradient for every choice of H, then $JNJN^T = \mu I$, where μ = constant ($\neq 0$).

Proof: Let $A \equiv JNJN^T$. Then $A \frac{\partial H}{\partial Q}$ is a gradient. First, consider the 2n classes of Hamiltonian functions which are polynomials in only one $Q_i \in \{Q_1,\ldots,Q_{2n}\}$. Then, the following vectors are gradients:

$$\begin{bmatrix} a_{1,1} & g_1(Q_1) \\ a_{2,1} & g_1(Q_1) \\ & \vdots & \\ a_{2n,1} & g_1(Q_1) \end{bmatrix}, \begin{bmatrix} a_{1,2} & g_2(Q_2) \\ a_{2,2} & g_2(Q_2) \\ & \vdots & \\ a_{2n,2} & g_2(Q_2) \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,2n} & g_{2n}(Q_{2n}) \\ a_{2,2n} & g_{2n}(Q_{2n}) \\ & \vdots & \\ a_{2n,2n} & g_{2n}(Q_{2n}) \end{bmatrix} \quad \text{(A.19)}$$

where

$$A \equiv \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2n} \\ \vdots & \vdots & & \vdots \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n} \end{bmatrix}$$

and

$$g_i(Q_i) \equiv \frac{\partial H(Q_i)}{\partial Q_i} \cdot \qquad (i = 1,\ldots,2n)$$

Since each of the vectors of Equations (A.19) is a gradient, there exist 2n functions $B_1(Q),\ldots,B_{2n}(Q)$ such that:

$$\begin{bmatrix} \frac{\partial B_i}{\partial Q_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial B_i}{\partial Q_{2n}} \end{bmatrix} = \begin{bmatrix} a_{1,i} & g_i(Q_i) \\ & \cdot \\ & \cdot \\ & \cdot \\ a_{2n,i} & g_i(Q_i) \end{bmatrix} . \quad (i = 1,\ldots,2n) \quad (A.20)$$

Given a particular $i \in \{1,2,\ldots,2n\}$, Equation (A.20) can be viewed as an integrable system of first-order partial differential equations with dependent variable B_i. Thus, the integrability conditions must be satisfied (i.e., $\frac{\partial^2 B_i}{\partial Q_j \partial Q_k} = \frac{\partial^2 B_i}{\partial Q_k \partial Q_j}$). This is equivalent to the requirement that

$$\frac{\partial}{\partial Q} \begin{bmatrix} a_{1,i} & g_i(Q_i) \\ & \cdot \\ & \cdot \\ & \cdot \\ a_{2n,i} & g_i(Q_i) \end{bmatrix} \qquad (i = 1,\ldots,2n)$$

be a symmetric matrix. Thus,

$$\frac{\partial(a_{i,k} g_k)}{\partial Q_j} = \frac{\partial(a_{j,k} g_k)}{\partial Q_i}$$

for each $i, j, k = 1,2,\ldots,2n$. Consider the case $i = k \neq j$:

$$\frac{\partial a_{i,k}}{\partial Q_j} g_k + a_{i,k} \frac{\partial g_k}{\partial Q_j} = \frac{\partial a_{j,k}}{\partial Q_i} g_k + a_{j,k} \frac{\partial g_k}{\partial Q_i} .$$

But, g_k depends on only Q_k and since $i = k$, $j \neq k$

$$\frac{\partial a_{k,k}}{\partial Q_j} g_k + 0 = \frac{\partial a_{j,k}}{\partial Q_k} g_k + a_{j,k} \frac{\partial g_k}{\partial Q_k} . \qquad (A.21)$$

Suppose $H(Q_k)$ is a first-degree polynomial in Q_k. Then,

$$g_k \equiv \frac{\partial H}{\partial Q_k} = \text{constant} \rightarrow \frac{\partial g_k}{\partial Q_k} = 0 .$$

Since Equation (A.21) must hold for all choices of H, it follows that

$$\frac{\partial a_{k,k}}{\partial Q_j} g_k = \frac{\partial a_{j,k}}{\partial Q_k} g_k$$

or,

$$\left(\frac{\partial a_{k,k}}{\partial Q_j} - \frac{\partial a_{j,k}}{\partial Q_k} \right) g_k = 0 .$$

Since $g_k \neq 0$ in general, then

$$\frac{\partial a_{k,k}}{\partial Q_j} = \frac{\partial a_{j,k}}{\partial Q_k} . \qquad (j \neq k) \tag{A.22}$$

Substitution of Equation (A.22) into Equation (A.21) then shows that

$$a_{j,k} = 0 \qquad (j \neq k) \tag{A.23}$$

since $\frac{\partial g_k}{\partial Q_k} \neq 0$, in general.

From Equation (A.23) it follows that A must be a diagonal matrix, and then Equation (A.22) becomes

$$\frac{\partial a_{k,k}}{\partial Q_j} = 0 , \qquad (j \neq k)$$

which implies either $a_{k,k} = a_{k,k}(Q_k)$ or $a_{k,k} = \text{constant}$.

Finally, consider the class of Hamiltonian functions $H = Q_i Q_{i+1}$ $(i = 1,\ldots,2n-1)$. Then the following vectors are gradients:

$$\begin{bmatrix} a_{1,1}(Q_1)Q_2 \\ a_{2,2}(Q_2)Q_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_{2,2}(Q_2)Q_3 \\ a_{3,3}(Q_3)Q_2 \\ 0 \\ \cdot \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ a_{2n-1,2n-1}(Q_{2n-1})Q_{2n} \\ a_{2n,2n}(Q_{2n})Q_{2n-1} \end{bmatrix} \qquad (A.24)$$

Again Equations (A.21) must be satisfied, so operating on Equations (A.24)

$$\frac{\partial[a_{1,1}(Q_1)Q_2]}{\partial Q_2} = \frac{\partial[a_{2,2}(Q_2)Q_1]}{\partial Q_1}$$

$$\frac{\partial[a_{2,2}(Q_2)Q_3]}{\partial Q_3} = \frac{\partial[a_{3,3}(Q_3)Q_2]}{\partial Q_2}$$

$$\vdots \qquad \qquad \vdots$$

$$\frac{\partial[a_{2n-1,2n-1}(Q_{2n-1})Q_{2n}]}{\partial Q_{2n}} = \frac{\partial[a_{2n,2n}(Q_{2n})Q_{2n-1}]}{\partial Q_{2n-1}}$$

These conditions imply that

$$a_{1,1}(Q_1) = a_{2,2}(Q_2)$$

$$a_{2,2}(Q_2) = a_{3,3}(Q_3)$$

$$\vdots \qquad \qquad \vdots$$

$$a_{2n-1,2n-1}(Q_{2n-1}) = a_{2n,2n}(Q_{2n}) .$$

Thus, $a_{1,1}(Q_1) = a_{2,2}(Q_2) = \ldots = a_{2n,2n}(Q_{2n})$. But the set $\{Q_1, Q_2, \ldots, Q_{2n}\}$ is independent, so each of the diagonal elements must be the same non-zero constant, i.e.,

$$a_{1,1} = a_{2,2} = \ldots = a_{2n,2n} \equiv \mu^* = \text{constant} .$$

Thus, from the above lemma

$$JNJN^T = \mu^* I ,$$

or

$$NJN^T = \mu^* J^{-1} = -\mu^* J \equiv \mu J$$

Therefore, N is symplectic and the theorem is proved.

Property 2.1: Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ matrix, where A , B , C , and D are $n \times n$ submatrices. Then, M is a symplectic matrix if and only if A^TC and B^TD are symmetric and $D^TA - B^TC = \mu I$.

Proof: ($\rightarrow$) To show that A^TC and B^TD are symmetric, it must be proved that $A^TC = (A^TC)^T$ and $B^TD = (B^TD)^T$. It is known that $M^TJM = \mu J$ by hypothesis, and it is easily verified that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} .$$

Thus,

$$M^TJM = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

or,

$$M^TJM = \begin{bmatrix} -C^TA + A^TC & -C^TB + A^TD \\ -D^TA + B^TC & -D^TB + B^TD \end{bmatrix}. \tag{A.25}$$

But, M is symplectic, so

$$-C^TA + A^TC = 0$$

$$-D^TB + B^TD = 0$$

$$-D^TA + B^TC = -\mu I$$

as desired.

($\leftarrow$) Conversely, let $A^TC = C^TA$, $B^TD = D^TB$, and $D^TA - B^TC = \mu I$ be given. From Equation (A.25)

$$M^TJM = \begin{bmatrix} C^TA + A^TC & C^TB + A^TD \\ -D^TA + B^TC & -D^TB + B^TD \end{bmatrix}$$

or,

$$M^TJM = \begin{bmatrix} 0 & -C^TB + A^TD \\ -\mu I & 0 \end{bmatrix}. \tag{A.26}$$

To show M is symplectic, then, it needs to be shown that $-C^TB + A^TD = \mu I$. Consider the transpose of the hypothesis $D^TA - B^TC = \mu I$:

$$(D^TA - B^TC)^T = (\mu I)^T$$

or,

$$A^T D - C^T B = \mu I$$

as desired, and Property (2.1) is verified.

APPENDIX B

In dealing with the base solutions of Chapters 3 and 4, the following Taylor series expansions are useful for low-eccentricity missions.

$$\tan^{-1}\{[(1 - q)\tan\frac{\theta}{2} + s]\,(1 - q^2 - s^2)^{-\frac{1}{2}}\} \simeq \frac{1}{2}\,[\theta$$

$$- q\sin\theta + s(\cos\theta + 1) + \frac{1}{2}(q^2 - s^2)\sin\theta\cos\theta$$

$$+ qs\sin^2\theta + (q^3/3 - 3qs^2/4)\sin^3\theta$$

$$+ (s^3/3 - q^2 s)\,(\cos\theta - 1)\sin^2\theta - \frac{1}{2}q^3\sin\theta$$

$$- \frac{1}{4}(s^3 + q^2 s)\,(\cos\theta - 1)^2] + O(q^4, s^4) \tag{B.1}$$

$$\frac{1}{1 - x} = 1 + x + x^2 + \ldots + x^n + \ldots \qquad (x \in \{q, s\}) \tag{B.2}$$

$$\frac{1}{\gamma} = 1 - (q\cos\theta + s\sin\theta) + - \ldots + (-1)^n\,(q\cos\theta + s\sin\theta)^n + \ldots \tag{B.3}$$

$$\frac{1}{\gamma^2} = 1 - 2(q\cos\theta + s\sin\theta) + - \ldots$$

$$+ (-1)^n\,(n + 1)\,(q\cos\theta + s\sin\theta)^n + \ldots, \tag{B.4}$$

where $\gamma \equiv 1 + q\cos\theta + s\sin\theta$.

$$\frac{1}{(1 - q^2 - s^2)} = 1 + q^2 + s^2 + O(q^4, s^4) \tag{B.5}$$

$$\frac{1}{(1 - q^2 - s^2)^{3/2}} = 1 + \frac{3}{2}(q^2 + s^2) + O(q^4, s^4)\,. \tag{B.6}$$

APPENDIX C

The initial conditions, terminal conditions, and parameter values used in the numerical studies of Chapter 7 are presented below.

Escape Mission:

$k = 3.9860640 \times 10^{14}$ meters^3/seconds^2

$I_{sp} = 5000$ seconds

$r_o = 6.6781700 \times 10^6$ meters

$\theta_o = 0$

$\dot{r}_o = 0$

$r_o\dot{\theta}_o = 7.7257984 \times 10^4$ meters/second

$W_o = 10{,}000$ pounds

Circular Orbit Transfer:

$k = 3.9860640 \times 10^{14}$ meters^3/seconds^2

$I_{sp} = 5000$ seconds

$r_o = 6.7002574 \times 10^6$ meters

$\theta_o = 0$

$\dot{r}_o = 0$

$r_o\dot{\theta}_o = 7.7130543 \times 10^4$ meters/second

$W_o = 10{,}000$ pounds

$r_f = 4.2266831 \times 10^7$ meters

$\dot{r}_f = 0$

$r_f\dot{\theta}_f = 3.0709467 \times 10^4$ meters/second .

BIBLIOGRAPHY

1. Abraham, R., Foundations of Mechanics, W. A. Benjamin, Inc., New York, 1967.

2. Bellman, R. E., Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

3. Bliss, G. A., Lectures on the Calculus of Variations, The University of Chicago Press, Chicago, 1946.

4. Boeing Company, "Trajectory Optimization and Guidance for the Earth Orbit Rendezvous Mission," (3 Volumes), Technical Report D5-13136, July 1, 1965.

5. Born, M., The Mechanics of the Atom, Frederick Ungar Publishing Co., New York, 1960.

6. Caratheodory, C., Variationsrechnung und Partielle Differentialgleichungen Erste Ordnung, B. G. Teubner, Berlin, 1935.

7. Caratheodory, C., Calculus of Variations and Partial Differential Equations of the First Order, (translated by R. B. Dean and J. J. Brandstatter), Holden-Day, Inc., San Francisco, Part I, 1965; Part II, 1967.

8. DeZur, R. S., and G. W. Haynes, "Contact Transformations and the Theory of Optimal Control," NASA Contractor Report CR-306, September 1965.

9. Eckenwiler, M. W., "Closed-Form Lagrangian Multipliers for Coast Periods of Optimum Trajectories," AIAA Journal, June 1965, pp. 1149-1151.

10. Fraeijs de Veubeke, B., "Canonical Transformations and the Thrust-Coast-Thrust Optimal Transfer Problem," Astronautica Acta, July-August 1965, pp. 271-282.

11. Fraeijs de Veubeke, B., "Optimal Steering and Cutoff-Relight Programs for Orbital Transfers," Astronautica Acta, July-August 1966, pp. 323-328.

12. Hamilton, W. R., "Second Essay on a General Method in Dynamics," Philosophical Transactions of the Royal Society of London, 1835, p. 98.

13. Hempel, P. R., "Representation of the Lagrangian Multipliers for Coast Periods of Optimum Trajectories," AIAA Journal, April 1966, pp. 729-730.

14. Hermes, H., "Investigation of Sufficiency Conditions and the Hamilton-Jacobi Approach to Optimal Control Problems," NASA Technical Memorandum TM X-53150, October 1964, pp. 49-119.

15. Hinz, H. K., "Optimal Low-Thrust Near-Circular Orbit Transfer," AIAA Journal, June 1963, pp. 1367-1371.

16. Irving, J. H., Space Technology, John Wiley and Sons, Inc., New York, 1959, Chapter 10.

17. Jacobi, C. G. J., Vorlesungen Uber Dynamik, Druck und Verlag von G. Reimer, Berlin, 1884.

18. Johnson, D. P. and L. W. Stumpf, "Perturbation Solutions for Low-Thrust Rocket Trajectories," AIAA Journal, October 1965, pp. 1934-1936.

19. Kalman, R. E., "Contributions to the Theory of Optimal Control," Boletin De La Sociedad Mathematica Mexicana, Second Series, Vol.5, 1960, pp. 102-119.

20. Kalman, R. E. Mathematical Optimization Techniques, University of California Press, Berkeley, California, 1963, Chapter 16.

21. Lagrange, J. L., Mecanique Analytique (Fourth Edition), Gauthier-Villars and Sons, Paris, 1888 (First Edition published in 1811).

22. Lewallen, J. M., and B. D. Tapley, "Analysis and Comparison of Several Numerical Optimization Methods," AIAA Aerospace Sciences Meeting, New York, January 1967.

23. Lie, S. and G. Scheffers, Geometrie der Berührungstransformationen, B. G. Teubner, Liepzig, 1896.

24. Melbourne, W. G., "Interplanetary Trajectory and Payload Capability of Advanced Propulsion Vehicles," JPL Technical Report TR 32-68, March 1961.

25. Miner, W. E., "Transformation of the λ-Vector and Closed Form Solutions of the λ-Vector on Coast Arcs," NASA-MSFC Aeroballistics Internal Note No. 20-63, April 1963.

26. Mitchell, "High Thrust Optimization Through Hamilton-Jacobi Theory," M.S. Thesis, Massachusetts Institute of Technology, June 1966.

27. Moeckel, W. E., "Trajectories with Constant Tangential Thrust in Central Gravitational Fields," NASA Technical Report TR R-53, 1960.

28. Nafoosi, A. A. and H. Passmore, "On an Application of the Hamilton-Jacobi Theory to High-Thrust Rocket Flight," NASA Technical Memorandum TM X-53292, July 1965, pp. 267-296.

29. Ng, C. H. and P. J. Palmadesso, "Analytical Solution of Euler-Lagrange Equations for Optimum Coast Trajectories," NASA Technical Memorandum TM X-53478, June 1966, pp. 131-140.

30. Perkins, F. M., "Flight Mechanics of Low-Thrust Spacecraft," Journal of the Aero/Space Sciences, May 1959, pp. 291-297.

31. Pinkham, G., "An Application of a Successive Approximation Scheme to Optimizing Very Low-Thrust Trajectories," NASA-MSFC Report MTP-AERO-63-12, February 1963, pp. 57-62.

32. Pitkin, E. T., "Low-Thrust Trajectories -- A Bibliography," Journal of the Astronautical Sciences, January-February 1966, pp. 21-28.

33. Poincare, H., Les Methodes Nouvelles de la Mecanique Celeste (Volume I), Dover Publications Inc., New York, 1957.

34. Pontryagin, L. S., V. G. Boltyanski, R. V. Gamkrelidze, and E. F. Mischenko, The Mathematical Theory of Optimal Processes, Interscience Publication, John Wiley and Sons, Inc., New York, 1962.

35. Powers, W. F., "The Effects of a Coast Period in Determining a Space Rendezvous Trajectory," presented at the 14th Annual AIAA South-eastern Regional Student Conference, Atlanta, Georgia, April 1963.

36. Powers, W. F., "Hamiltonian Perturbation Theory for Optimal Trajectory Analysis," The University of Texas Engineering Mechanics Research Lab Report EMRL TR-1003, June 1966.

37. Powers, W. F. and B. D. Tapley, "Canonical Transformation Theory and the Optimal Trajectory Problem," The University of Texas Engineering Mechanics Research Lab Report EMRL TR-1022, August 1967.

38. Rund, H., The Hamilton-Jacobi Theory in the Calculus of Variations, D. Van Nostrand Co. Inc., Princeton, New Jersey, 1966.

39. Shi, Y. Y. and M. C. Eckstein, "Ascent or Descent from Satellite Orbit by Low Thrust," AIAA Journal, December 1966, pp. 2203-2209.

40. Siegel, C. L., Vorlesungen Uber Himmelsmechanik, Springer-Verlag, Berlin, 1956.

41. Sneddon, I. N., Elements of Partial Differential Equations, McGraw-Hill Book Co., Inc., New York, 1957.

42. Snow, D. R., "Caratheodory-Hamilton-Jacobi Theory in Optimal Control," Journal of Mathematical Analysis and Applications, Volume 17 (1967), pp. 99-118.

43. Stancil, R. T. and L. J. Kulakowski, "Rocket Boost Trajectories for Maximum Burnout Velocity," ARS Journal, July 1960, pp. 612-618.

44. Tsien, H. S., "Take-Off From a Satellite Orbit," ARS Journal, July-August 1953, pp. 233-236.

45. Valentine, F. A., "The Problem of Lagrange with Differential Inequalities as Added Side Conditions," Contributions to the Calculus of Variations 1933-1937, University of Chicago Press, Chicago, 1937, pp. 407-448.

46. Whittaker, E. T., Analytical Dynamics of Particles and Rigid Bodies (Fourth Edition), Cambridge University Press, London, 1964.

47. Wintner, A., The Analytical Foundations of Celestial Mechanics, The Princeton University Press, Princeton, New Jersey, 1941.